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NOTES ON THE THEORY OF ASSOCIATION OF ATTRIBUTES IN STATISTICS.

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THE simplest possible form of statistical classification is "division" (as the logicians term it) "by dichotomy," i.e. the sorting of the objects or individuals observed into one or other of two mutually exclusive classes according as they do or do not possess some character or *attribute*; as one may divide men into sane and insane, the members of a species of plants into hairy and glabrous, or the members of a race of animals into males and females. The mere fact that we do employ such a classification in any case must not of course be held to imply a natural and clearly defined boundary between the two classes; e.g. sanity and insanity, hairiness and glabrousness, may pass into each other by such fine gradations that judgments may differ as to the class in which a given individual should be entered. The judgment must however be finally *decisive*; intermediates not being classed as such even when observed.

The theory of statistics of this kind is of a good deal of importance, not merely because they are of a fairly common type—the statistics of hybridisation experiments given by the followers of Mendel may be cited as recent examples—but because the ideas and conceptions required in such theory form a useful introduction to the more complex and less purely logical theory of variables. The classical writings on the subject are those of De Morgan*, Boole† and Jevons‡, the method and notation of the latter being used in the following Notes, the first three sections of which are an abstract of the two memoirs referred to below§.

* *Formal Logic*, chap. VIII., "On the Numerically Definite Syllogism," 1847.

† *Analysis of Logic*, 1847. *Laws of Thought*, 1854.

‡ "On a General System of Numerically Definite Reasoning," *Memoirs of Manchester Literary and Philosophical Society*, 1870. Reprinted in *Pure Logic and other Minor Works*, Macmillan, 1890.

§ "On the Association of Attributes in Statistics," *Phil. Trans.* A, Vol. 194 (1900), p. 257. "On the theory of Consistence of Logical Class Frequencies," *Phil. Trans.* A, Vol. 197 (1901), p. 91.

1. *Notation; terminology; relations between the class frequencies; tabulation.*

The notation used is as follows*:

N = total number of observations,
(A) = no. of objects or individuals possessing attribute A,
(α) = „ „ *not* possessing attribute A,
(AB) = „ „ possessing both attributes A and B,
$(A\beta)$ = „ „ „ attribute A but not B.
(αB) = „ „ „ attribute B but not A,
$(\alpha\beta)$ = „ „ not possessing either attribute A or B,

and so on for as many attributes as are specified. A class specified by n attributes in this notation may be termed a class of the nth order. The attributes denoted by English capitals may be termed *positive* attributes, and their *contraries*, denoted by the Greek letters, *negative* attributes. If two classes are such that every attribute in the one is the negative or contrary of the corresponding attribute in the other they may be termed contrary classes, and their frequencies contrary frequencies; (AB) and $(\alpha\beta)$, $(AB\gamma)$ and $(\alpha\beta C)$ are for instance pairs of contraries.

If the complete series of frequencies arrived at by noting n attributes is being tabulated, frequencies of the same order should be kept together. Those of the same order are best arranged by taking separately the set or "aggregate" of frequencies, derivable from each positive class by substituting negatives for one or more of the positive attributes. Thus the frequencies for the case of three attributes may conveniently be tabulated in the order—

$$\left.\begin{array}{ll} \text{Order 0.} & N \\ \text{Order 1.} & (A), (\alpha) : (B), (\beta) : (C), (\gamma) \\ \text{Order 2.} & (AB), (A\beta), (\alpha B), (\alpha\beta) : (AC), (A\gamma), (\alpha C), (\alpha\gamma) : (BC), (B\gamma), (\beta C), (\beta\gamma) \\ \text{Order 3.} & (ABC), (\alpha BC), (A\beta C), (AB\gamma), (\alpha\beta C), (\alpha B\gamma), (A\beta\gamma), (\alpha\beta\gamma) \end{array}\right\} \quad (1).$$

But since all frequencies are used non-exclusively, (A) denoting the frequency of objects possessing the attribute A with or without others and so forth, the frequency of any class can always be expressed in terms of the frequencies of classes of higher order; that is to say we have

$$\left.\begin{array}{l} N = (A) + (\alpha) = (B) + (\beta) = \text{etc.} \\ \quad = (AB) + (A\beta) + (\alpha B) + (\alpha\beta) = \text{etc.} \\ (A) = (AB) + (A\beta) \\ \quad = (ABC) + (AB\gamma) + (A\beta C) + (A\beta\gamma) = \text{etc.} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

* I have substituted small Greek letters for Jevons' italics. Italics are rather troublesome when speaking, as one has to spell out a group like *AbcDE*, "big *A*, little *b*, little *c*, big *D*, big *E*." It is simpler to say $A\beta\gamma DE$. The Greek becomes more troublesome when many letters are wanted, owing to the non-correspondence of the alphabets, but this is not often of consequence.

Hence it is quite *unnecessary* to state *all* the frequencies as under (1); if space is of importance no more need be given than the eight frequencies of the third order, the *ultimate* frequencies as they may be termed (i.e. frequencies of classes specified by the whole number of attributes noted). The number of frequencies in an aggregate of order n, is evidently 2^n, so that not more than 2^n frequencies need be stated in any case where n attributes are observed.

Any other set of 2^n independent frequencies may however be chosen instead of the 2^n ultimate frequencies, the set formed by taking N together with the frequencies of all the positive classes offering several advantages. It is not difficult to see that any frequency whatever can be expressed in terms of the number of observations and the positive-class frequencies by using the relations (2). We have for instance

$$\left.\begin{aligned}
(\alpha) &= N-(A)\\
(\alpha B) &= (B)-(AB)\\
(\alpha\beta) &= (\alpha)-(\alpha B) = N-(A)-(B)+(AB)\\
(\alpha\beta\gamma) &= (\alpha\beta)-(\alpha\beta C) = (\alpha\beta)-(\alpha C)+(\alpha BC)\\
&= N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC)
\end{aligned}\right\} \quad \ldots\ldots(3),$$

and so on.

To take a very simple example with two attributes only, consider the statement of results of one of Mr Bateson's experiments on hybridisation with poultry, Leghorn-Dorking hybrids when crossed *inter se* produced offspring of varied forms; some having the rose comb and some not, some having the extra toe that characterises the Dorking and some not. Using A to denote "rose comb," B to denote extra toe, the numerical results may be completely expressed in either of the following forms

I	II
$(AB) = 208$	$N = 336$
$(A\beta) = 63$	$(A) = 271$
$(\alpha B) = 54$	$(B) = 262$
$(\alpha\beta) = 11$	$(AB) = 208.$

The advantages of the second form of tabulation are obvious; it gives at sight the whole number of observations and the numbers of A's and B's. The first table gives neither without reckoning, yet both are equally complete.

A rather interesting case arises where the frequencies of contrary classes are equal, as may be the case if the character dealt with is really variable and the points of division between A's and α's are taken at the medians. Such a condition implies necessary relations between the class-frequencies of any odd order and the frequencies of next lower order, but for the discussion of the case I must refer to the first memoir mentioned in my note § on p. 121.

2. *Consistence and Inference.*

Although the positive-class frequencies (including N under that heading) are all independent in the sense that no single one can be expressed in terms of the others, they are nevertheless subject to certain limiting conditions if they are to be self-consistent, i.e. such as might have been observed in one and the same field of observation or "universe," to use the convenient term of the logicians. Consider the case of three attributes, for example. It is evident that we must have

$$\left.\begin{array}{lll} (AB) \not< 0 & \text{as } (AB) \text{ must not be negative} \\ \not< (A)+(B)-N & \text{as } (\alpha\beta) \quad \text{,,} \quad \text{,,} \\ \not> (A) & \text{as } (A\beta) \quad \text{,,} \quad \text{,,} \\ \not> (B) & \text{as } (\alpha B) \quad \text{,,} \quad \text{,,} \end{array}\right\} \ldots\ldots\ldots (4);$$

and similar conditions must hold for (AC) and (BC). But these are not the only conditions that must hold. The second-order frequencies must not only be such as not to imply negative values for the frequencies of other classes of their own aggregates, but also *must not imply negative values for any of the third-order frequencies.* Expanding all the third-order frequencies in terms of the frequencies of positive classes, and putting the resulting expansion $\not< 0$, we have

	or the frequency given below will be negative	
$(ABC) \not< 0$	(ABC)	[1]
$\not< (AB)+(AC)-(A)$	$(A\beta\gamma)$	[2]
$\not< (AB)+(BC)-(B)$	$(\alpha B\gamma)$	[3]
$\not< (AC)+(BC)-(C)$	$(\alpha\beta C)$	[4]
$\not> (AB)$	$(AB\gamma)$	[5]
$\not> (AC)$	$(A\beta C)$	[6]
$\not> (BC)$	(αBC)	[7]
$\not> (AB)+(AC)+(BC)-(A)-(B)-(C)+N$	$(\alpha\beta\gamma)$	[8]

... (5).

But if any one of the minor limits [1]—[4] be greater than any one of the major limits [5]—[8] these conditions are impossible of fulfilment. There are four minor limits to be compared with four major limits or sixteen comparisons in all to be made; but the majority of these, twelve in all, only lead back to conditions of the form (4). The four comparisons of expansions due to contrary frequencies alone lead to new conditions—viz.

$$\left.\begin{array}{l} (AB)+(AC)+(BC) \not< (A)+(B)+(C)-N \\ (AB)+(AC)-(BC) \not> (A) \\ (AB)-(AC)+(BC) \not> (B) \\ -(AB)+(AC)+(BC) \not> (C) \end{array}\right\} \ldots\ldots\ldots\ldots (6).$$

These conditions give limits to any one of the three frequencies (AB), (AC) and (BC) in terms of the other two and the frequencies of the first order, i.e. enable us to infer limits to the one class-frequency in terms of the others. It will very usually happen in practical statistical cases that the limits so obtained are valueless, lying outside those given by the simpler conditions (4), but that is merely because in practice the values of the assigned frequencies, e.g. (AB) and (AC), seldom approach sufficiently closely to their limiting values to render inference possible.

3. *Association.*

Two attributes, A and B, are usually defined to be independent, within any given field of observation or "universe," when the chance of finding them together is the product of the chances of finding either of them separately. The physical meaning of the definition seems rather clearer in a different form of statement, viz. if we define A and B to be independent *when the proportion of A's amongst the B's of the given universe is the same as in that universe at large.* If for instance the question were put "What is the test for independence of small-pox attack and vaccination?", the natural reply would be "The percentage of vaccinated amongst the attacked should be the same as in the general population" or "The percentage of attacked amongst the vaccinated should be the same as in the general population." The two definitions are of course identical in effect, and permit of the same simple symbolical expression in our notation; the criterion of independence of A and B is in fact

$$(AB) = \frac{(A)(B)}{N} \quad \text{.....(7).}$$

In this equation the attributes specifying the universe are understood, not expressed. If all objects or individuals in the universe are to possess an attribute or series of attributes K it may be written

$$(ABK) = \frac{(AK)(BK)}{(K)}.$$

An equation of such form must be recognised as the criterion of independence for A and B within the universe K. As I have shewn in the first memoir referred to in note §, p. 121, if the relation (7) hold good, the three similar relations for the remaining frequencies of the "aggregate"—i.e. the set of frequencies obtained by substituting their contraries α, β for A or B or both—must also hold, viz.

$$\left.\begin{aligned} (\alpha B) &= \frac{(\alpha)(B)}{N} \\ (A\beta) &= \frac{(A)(\beta)}{N} \\ (\alpha\beta) &= \frac{(\alpha)(\beta)}{N} \end{aligned}\right\} \quad \text{.....(8),}$$

for we have at once

$$(\alpha B) = (B) - (AB) = \frac{(B)\{N-(A)\}}{N} = \frac{(B)(\alpha)}{N},$$

and so on. The case of two attributes is thus quite simple; the definition of independence is almost intuitive, and the criterion need only be applied to one frequency of the aggregate. The two attributes are termed positively or negatively *associated* according as (AB) is greater or less than the value it would have in the case of independence, or, to put the same thing another way, according as $(AB)/(A)$ is greater or less than $(B)/N$ or $(AB)/(B)$ greater or less than $(A)/N$.

If more than two attributes are noted in the record they must be dealt with, in the first instance, pair by pair as above, but subsequently the association between each pair should be observed in the sections of material or sub-universes defined by the other attributes. In the case of three attributes, for instance, we have not only to deal with the association between A and B in the universe at large but also in the universe of C's and in the universe of γ's—associations which will be tested by comparing $(ABC)/(AC)$ with $(BC)/(C)$ or $(ABC)/(BC)$ with $(AC)/(C)$, and $(AB\gamma)/(A\gamma)$ with $(B\gamma)/(\gamma)$ and so on. Such "partial" associations are of great practical importance, as a test for the correctness or otherwise of physical interpretations placed on any "total" associations observed. When A and B are found to be associated it is a common form of argument to say that the association is not "direct" but due to the association of A with C and B with C; the argument may be tested at once by finding whether A and B are still associated in the separated universes C and γ. If they are the argument is baseless. It has been said for instance that the association between vaccination and protection from small-pox is due to the association of both with sanitary conditions (a larger proportion of the upper classes than of the lower classes being vaccinated, according to the argument). To test the argument the "universe" or material observed should be limited *either* to those living wholly in sanitary conditions *or* to those living wholly in insanitary conditions (it does not matter which). "Partial" associations again are of importance to the biologist in the theory of heredity. If an attribute be heritable, its presence in the parent or the grandparent is associated with its presence in the offspring; but the physical interpretation to be placed on such inheritance depends very largely on whether there is also a *partial* inheritance from the grandparent, the presence of the attribute in the grandparent being associated with its presence in the offspring even when the parents either all possess or all do not possess the attribute.

It is important to notice that the test for association is necessarily based on a *comparison* of percentages or proportions, e.g. $(AB)/(A)$ and $(B)/N$. The mere fact that a certain number of A's are B gives no physical information; besides knowing how many A's are B you must know also how many non-A's are B or what proportion of A's exists in the given universe at large. In an investigation as to the causation of A it is therefore just as important to observe non-A's as A's.

This point is frequently forgotten. In an investigation as to the inheritance of deaf-mutism in America*, for instance, only the offspring of deaf-mutes were observed, and the argument consequently breaks down on page after page into conjectural statements as to points on which the editor has no information—e.g. the proportion of deaf-mutes amongst the children of normals.

The difference of $(AB)/(A)$ from $(B)/N$ and of $(AB)/(B)$ from $(A)/N$ are of course not, as a rule, the same, and it would be useful and convenient to measure the "association" by some more symmetrical method—a "coefficient of association" ranging between ± 1 like the coefficient of correlation. In the first memoir referred to in note §, p. 121, such a coefficient, of empirical form, was suggested, but that portion of the memoir should now be read in connection with a later memoir by Professor Pearson †.

4. *On the theory of complete independence of a series of Attributes.*

The tests for independence are by no means simple when the number of attributes is more than two. Under what circumstances should we say that a series of attributes $ABCD\ldots$ were completely independent? I believe not a few statisticians would reply at once "if the chance of finding them together were equal to the product of the chances of finding them separately," yet such a reply would be in error. The mere result

$$\frac{(ABCD\ldots)}{N} = \frac{(A)}{N} \cdot \frac{(B)}{N} \cdot \frac{(C)}{N} \cdot \frac{(D)}{N} \cdots\cdots \qquad \ldots\ldots\ldots\ldots (9)$$

does not in general give any information as to the independence or otherwise of the attributes concerned. If the attributes are *known to be* completely independent then certainly the relation (9) holds good, but the converse is not true. "Equations of independence" of the form (9) must be shewn to hold for more than one class of any aggregate, of an order higher than the second, before the complete independence of the attributes can be inferred.

From the physical point of view complete independence can only be said to subsist for a series of attributes $ABCD\ldots$ within a given universe, when every pair of such attributes exhibits independence not only within the universe at large but also in every sub-universe specified by one or more of the remaining attributes of the series, or their contraries. Thus three attributes A, B, C are completely independent within a given universe if AB, AC and BC are independent within that universe and also

AB independent within the universes C and γ,
AC „ „ „ B „ β,
BC „ „ „ A „ α.

* *Marriages of the Deaf in America*, ed. by E. A. Fay. Volta Bureau, Washington, 1898.
† *Phil. Trans.* Vol. 195, p. 16.

If a series of attributes are completely independent according to this definition relations of the form (9) must hold for the frequency of every class of every possible order. Take the class-frequency $(ABCD)$ of the fourth order for instance. A and B are, by the terms of the definition, independent within the universe CD. Therefore

$$(ABCD) = \frac{(ACD)(BCD)}{(CD)}.$$

But A and C, and also B and C, are independent within the universe D. Therefore the fraction on the right is equal to

$$\frac{1}{(CD)} \cdot \frac{(AD)(CD)}{(D)} \cdot \frac{(BD)(CD)}{(D)} = \frac{(AD)(BD)(CD)}{(D)^2}.$$

But again AD, BD, CD are each independent within the universe at large; therefore finally

$$(ABCD) = \frac{1}{(D)^2} \cdot \frac{(A)(D)}{N} \cdot \frac{(B)(D)}{N} \cdot \frac{(C)(D)}{N} = \frac{(A)(B)(C)(D)}{N^3}.$$

Any other frequency can be reduced step by step in precisely the same way.

Now consider the converse problem. The total frequency N is given and also the n frequencies (A), (B), (C), etc. In how many of the ultimate frequencies $(ABCD \ldots MN)$, $(\alpha BCD \ldots MN)$, etc. must "relations of independence" of the form

$$(ABCD \ldots MN) = \frac{(A)(B)(C)(D) \ldots (M)(N)}{N^{n-1}}$$

hold good, in order that complete independence of the attributes may be inferred? The answer is suggested at once by the following consideration. The number of ultimate frequencies (frequencies of order n) is 2^n; the number of frequencies given is $\overline{n+1}$. If then all but $\overline{n+1}$ of the ultimate frequencies are given in terms of the equations of independence, the remaining frequencies are determinate; either these determinate values must be those that would be given by equations of independence, *or a state of complete independence must be impossible.* Suppose all the ultimate class-frequencies to have been tested and found to be given by the equations of independence, with the exception of the negative class $(\alpha\beta\gamma\delta \ldots \mu\nu)$ and the n classes with one positive attribute $(A\beta\gamma\delta \ldots \mu\nu)$, $(\alpha B\gamma\delta \ldots \mu\nu)$, etc. Take any one of these untested class-frequencies, $(A\beta\gamma\delta \ldots \mu\nu)$, and we have for example

$$\begin{aligned}(A\beta\gamma\delta \ldots \mu\nu) = (A) &- (ABCD \ldots MN) \\ &- (ABCD \ldots M\nu) \\ &- \text{other terms with one negative} \\ &- (ABCD \ldots \mu\nu) \\ &- \text{other terms with two negatives} \\ &- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &- (AB\gamma\delta \ldots \mu\nu) \\ &- \text{other terms with } \overline{n-2} \text{ negatives.}\end{aligned}$$

But all the frequencies on the right are given by the relations of independence. Therefore

$$(A\beta\gamma\delta\ldots\mu\nu) = \frac{(A)}{N^{n-1}}\{N^{n-1} - (B)(C)(D)\ldots(M)(N) - (B)(C)(D)\ldots(M)(\nu) - \ldots\ldots\ldots\ldots - (B)(C)(D)\ldots(\mu)(\nu) - \ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots - (B)(\gamma)(\delta)\ldots(\mu)(\nu) - \ldots\ldots\ldots\ldots\}.$$

Now consider the terms on the right in the bracket. With the exception of the one term $(B)(\gamma)(\delta)\ldots(\mu)(\nu)$, the remainder can be grouped in pairs of which the one member contains (B) and the other (β), the following frequencies in each member of the pair being the same. Carrying out this rearrangement the expression will read

$$(A\beta\gamma\delta\ldots\mu\nu) = \frac{(A)}{N^{n-1}}\{N^{n-1} - (B)(\gamma)(\delta)\ldots(\mu)(\nu) - (B)(C)(D)\ldots(M)(N) - (\beta)(C)(D)\ldots(M)(N) - (B)(C)(D)\ldots(M)(\nu) - (\beta)(C)(D)\ldots(M)(\nu) - \ldots\ldots\ldots\ldots - (B)(C)(\delta)\ldots(\mu)(\nu) - (\beta)(C)(\delta)\ldots(\mu)(\nu) - \ldots\ldots\ldots\ldots\}.$$

Replace (B) by $N-(\beta)$ throughout and rearrange the terms in similar pairs containing C and γ. (B) and (β) are then eliminated from all the terms and the expression then becomes

$$(A\beta\gamma\delta\ldots\mu\nu) = \frac{(A)}{N^{n-1}}\{N^{n-1} + (\beta)(\gamma)(\delta)\ldots(\mu)(\nu) - N(\gamma)(\delta)\ldots(\mu)(\nu) - N(C)(\delta)\ldots(\mu)(\nu) - \ldots\ldots\ldots\ldots - N(C)(D)\ldots(M)(N) - N(\gamma)(D)\ldots(M)(N) - \ldots\ldots\ldots\ldots - N(C)(D)\ldots(M)(\nu) - N(\gamma)(D)\ldots(M)(\nu) - \ldots\ldots\ldots\ldots\}.$$

Replacing C by $N-(\gamma)$ and regrouping in similar pairs of terms containing (D) and (δ) this will become

$$(A\beta\gamma\delta \dots \mu\nu) = \frac{(A)}{N^{n-1}} \{N^{n-1} + (\beta)(\gamma)(\delta)\dots(\mu)(\nu) - N^{n-2}(D)(E)\dots(M)(N) - N^{n-2}(\delta)(E)\dots(M)(N) - \text{etc.}\}$$

and continuing the same process until all the frequencies $(D)(E)\dots(M)(N)$ are eliminated, i.e. $\overline{n-1}$ times altogether,

$$(A\beta\gamma\delta \dots \mu\nu) = \frac{(A)(\beta)(\gamma)(\delta)\dots(\mu)(\nu)}{N^{n-1}}.$$

That is to say the theorem must be true quite generally: "A series of n attributes $ABC\dots MN$ are completely independent if the relations of independence are *proved* to hold for $(2^n - \overline{n+1})$ of the 2^n ultimate frequencies; such relations must then hold for the remaining $\overline{n+1}$ frequencies also." If the ultimate frequencies are only given by the relations of independence in n cases or less, independence may exist for certain pairs of attributes in certain universes but not in general. The mere fact of the relation holding for one class, e.g.

$$(ABCD\dots MN) = \frac{(A)(B)(C)(D)\dots(M)(N)}{N^{n-1}},$$

implies nothing—in striking contrast to the simple case of two attributes, where $2^n - \overline{n+1} = 1$ and only the one class-frequency need be tested in order to see if independence exists. In the case of three attributes the number of third-order classes is eight, of which four must be tested in order to be certain that complete independence exists. In the case of four attributes there are sixteen fourth-order classes of which eleven must be tested, and so on.

I have dealt with the problem hitherto on the assumption that only the first-order and the nth order frequencies were given, and that the frequencies of intermediate orders were unknown—or at least uncalculated, for of course the frequencies of all lower orders may be expressed in terms of those of the nth order. If however the frequencies of all orders may be supposed known, the above result may be thrown into a somewhat interesting form. It will be remembered that the frequency of any class of any order may be expressed in terms of the frequencies of the *positive* classes $[(A)(AB)(AC)(ABC)$ etc.] of its own and lower orders. Then *complete independence* exists for a series of attributes *if the criterion of independence hold for all the positive-class frequencies up to that of the nth order.* If we have for instance

$$(ABCD\dots MN) = \frac{1}{N^{n-1}}\{(A)(B)(C)(D)\dots(M)(N)\},$$

and also

$$(BCD \dots MN) = \frac{1}{N^{n-2}} \{(B)(C)(D) \dots (M)(N)\},$$

we must have

$$\begin{aligned}(\alpha BCD \dots MN) &= (BCD \dots MN) - (ABCD \dots MN) \\ &= \frac{1}{N^{n-1}} \{(B)(C)(D) \dots (M)(N)\} \{N - (A)\} \\ &= \frac{1}{N^{n-1}} (\alpha)(B)(C)(D) \dots (M)(N),\end{aligned}$$

and so on. The number of class-frequencies to be tested in order to demonstrate the existence of complete independence is, of course, the same as before, viz. $2^n - \overline{n+1}$.

It should be noted as a consequence of these results that the definition of "complete independence" given on p. 127 is redundant in its terms. It is quite true that if complete independence subsist for a series of attributes every possible pair must exhibit independence in every possible sub-universe as well as in the universe at large, but it is not necessary to apply the criterion of independence to *all* these possible cases. In the case of three attributes for instance the criterion of independence need only be applied to four frequencies, as we have just seen, in order to demonstrate complete independence; it cannot then be *necessary*, as suggested by the definition, to test nine different associations, viz.

$$\begin{array}{lll} |\, AB \,| & |\, AB \,|\, C \,| & |\, AB \,|\, \gamma \,|\,, \\ |\, AC \,| & |\, AC \,|\, B \,| & |\, AC \,|\, \beta \,|\,, \\ |\, BC \,| & |\, BC \,|\, A \,| & |\, BC \,|\, \alpha \,|\,, \end{array}$$

in the notation of my memoir on Association (an expression like $|\, AB \,|\, C \,|$ specifying "the association between A and B in the universe of C's"). It is in fact only necessary to test $|\, AB \,|$, $|\, AC \,|$, $|\, BC \,|$, and $|\, AB \,|\, C \,|$ (or one of the other three partial associations in positive universes). If these are zero, the remaining associations must be zero also; for we are given

$$(ABC) = \frac{1}{(C)}(AC)(BC) = \frac{1}{N^2}(A)(B)(C),$$

$$\begin{aligned}\therefore (ABC) &= \frac{1}{(B)}(AB)(BC) \\ &= \frac{1}{(A)}(AB)(AC) = \text{etc.}\end{aligned}$$

i.e. $|\, AC \,|\, B \,|$, $|\, BC \,|\, A \,|$, etc. are zero. Quite generally, it is only necessary, if the testing be supposed to proceed from the second order classes upwards, to test *one* of all the possible partial associations corresponding to each positive class. If there be four attributes $ABCD$, the six total associations $|\, AB \,|$, $|\, AC \,|$, $|\, AD \,|$, $|\, BC \,|$

etc. must first be tried; if they are zero, then follow on with $|AB|C|$, $|AB|D|$, $|AC|D|$ and $|BC|D|$, but not $|AC|B|$ or $|AD|B|$ etc.; if they are zero then finally try $|AB|CD|$, if it also be zero then the attributes are completely independent. It is not necessary to try further $|AC|BD|$ or $|AD|BC|$ etc.

The inadequacy of the usual treatment of independence arises from the fact that it proceeds wholly *à priori*, and generally has reference solely to cases of artificial chance. The result is an illusory appearance of simplicity. It is pointed out that if one "event" can "succeed" in a_1 and "fail" in b_1 ways, a second succeed in a_2 and fail in b_2 ways, and so on, the combined events can take place (succeed or fail) in

$$(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)$$

ways and succeed in

$$a_1a_2\dots a_n$$

ways. The chance of entire "success" is therefore

$$\frac{a_1a_2\dots a_n}{(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)},$$

the chance of the first event failing and the rest succeeding is

$$\frac{b_1a_2\dots a_n}{(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)},$$

and so on for all other possible cases. In short the chance of occurrence of the combined independent events is the product of the chances of the separate events. There the treatment stops, and all practical difficulties are avoided. In such text-book treatment it is *given* that the events are independent and *required* to deduce the consequences; in the problems that the statistician has to handle the consequences—the bare facts—are given and it is required to find whether the "events" or attributes are independent, wholly or in part.

5. *On the fallacies that may be caused by the mixing of distinct records.*

It follows from the preceding work that we cannot infer independence of a pair of attributes within a sub-universe from the fact of independence within the universe at large. From $|AB|=0$, we cannot infer $|AB|C|=0$ or $|AB|\gamma|=0$, although we can of course make the corresponding inference in the case of complete association—i.e. from $|AB|=1$ we do infer $|AB|C|=|AB|\gamma|=\text{etc.}=1$. But the converse theorem is also true; a pair of attributes does not necessarily exhibit independence within the universe at large even if it exhibit independence in *every* sub-universe; given

$$|AB|C|=0, \qquad |AB|\gamma|=0,$$

we cannot infer $|AB|=0$. The theorem is of considerable practical importance from its inverse application; i.e. even if $|AB|$ have a sensible positive or

negative value we cannot be sure that nevertheless $|AB|C|$ and $|AB|\gamma|$ are not both zero. Some given attribute might, for instance, be inherited neither in the male line nor the female line; yet a mixed record might exhibit a considerable apparent inheritance. Suppose for instance that 50 °/₀ of the fathers and of the sons exhibit the attribute, but only 10 °/₀ of the mothers and daughters. Then if there be no inheritance in either line of descent the record must give (approximately)

fathers	with	attribute	and sons	with	attribute	25 °/₀
„	„	„	„	without	„	25 °/₀
„	without	„	„	with	„	25 °/₀
„	„	„	„	without	„	25 °/₀
mothers	with	attribute	and daughters	with	attribute	1 °/₀
„	„	„	„	without	„	9 °/₀
„	without	„	„	with	„	9 °/₀
„	„	„	„	without	„	81 °/₀.

If these two records be mixed in equal proportions we get

parents	with	attribute	and offspring	with	attribute	13 °/₀
„	„	„	„	without	„	17 °/₀
„	without	„	„	with	„	17 °/₀
„	„	„	„	without	„	53 °/₀

Here $13/30 = 43\frac{1}{3}$ °/₀ of the offspring of parents with the attribute possess the attribute themselves, but only 30 °/₀ of offspring in general, i.e. there is quite a large but illusory inheritance created simply by the mixture of the two distinct records. A similar illusory association, that is to say an association to which the most obvious physical meaning must not be assigned, may very probably occur in any other case in which different records are pooled together or in which only one record is made of a lot of heterogeneous material.

Consider the case quite generally. Given that $|AB|C|$ and $|AB|\gamma|$ are both zero, find the value of (AB). From the data we have at once

$$(AB\gamma) = \frac{(A\gamma)(B\gamma)}{(\gamma)} = \frac{[(A)-(AC)][(B)-(BC)]}{[N-(C)]},$$

$$(ABC) = \frac{(AC)(BC)}{(C)}.$$

Adding

$$(AB) = \frac{N(AC)(BC)-(A)(C)(BC)-(B)(C)(AC)+(A)(B)(C)}{(C)[N-(C)]}.$$

Write

$$(AB)_0 = \frac{1}{N}(A)(B), \quad (AC)_0 = \frac{1}{N}(A)(C), \quad (BC)_0 = \frac{1}{N}(B)(C),$$

subtract $(AB)_0$ from both sides of the above equation, simplify, and we have

$$(AB) - (AB)_0 = \frac{N[(AC) - (AC)_0][(BC) - (BC)_0]}{C[N - (C)]}.$$

That is to say, *there will be apparent association between A and B in the universe at large unless either A or B is independent of C.* Thus, in the imaginary case of inheritance given above, if A and B stand for the presence of the attribute in the parents and the offspring respectively, and C for the male sex, we find a positive association between A and B in the universe at large (the pooled results) because A and B are both positively associated with C, i.e. the males of both generations possess the attribute more frequently than the females. The "parents with attribute" are mostly males; as we have only noted offspring of the same sex as the parents, their offspring must be mostly males in the same proportion, and therefore more liable to the attribute than the mostly-female offspring of "parents without attribute." It follows obviously that if we had found no inheritance to exist in any one of the *four* possible lines of descent (male-male, male-female, female-male, and female-female), no fictitious inheritance could have been introduced by the pooling of the *four* records. The pooling of the two records for the crossed-sex lines would give rise to a fictitious negative inheritance—disinheritance—cancelling the positive inheritance created by the pooling of the records for the same-sex lines. I leave it to the reader to verify these statements by following out the arithmetical example just given should he so desire.

The fallacy might lead to seriously misleading results in several cases where mixtures of the two sexes occur. Suppose for instance experiments were being made with some new antitoxin on patients of both sexes. There would nearly always be a difference between the case-rates of mortality for the two. If the female cases terminated fatally with the greater frequency and the antitoxin were administered most often to the males, a *fictitious* association between "antitoxin" and "cure" would be created at once. The general expression for $(AB) - (AB)_0$ shews how it may be avoided; it is only necessary to *administer the antitoxin to the same proportion of patients of both sexes.* This should be kept constantly in mind as an essential rule in such experiments if it is desired to make the most use of the results.

The fictitious association caused by mixing records finds its counterpart in the spurious correlation to which the same process may give rise in the case of continuous variables, a case to which attention was drawn and which was fully discussed by Professor Pearson in a recent memoir*. If two separate records, for each of which the correlation is zero, be pooled together, a spurious correlation will necessarily be created unless the mean of one of the variables, at least, be the same in the two cases.

* *Phil. Trans.* A, Vol. 192, p. 277.

A FURTHER STUDY OF STATISTICS RELATING TO VACCINATION AND SMALLPOX.

By W. R. MACDONELL, LL.D.

(1) The following paper is a continuation of the Note published in *Biometrika*, Vol. I. No. 3, but I have been able to extend my results to a comparison of the differential character of two epidemics in the same city, and to some preliminary consideration of the influence of occupation on the existence or non-existence of vaccination and on the severity of a smallpox attack. My statistical material is taken from: *A Summary of Statistics relating to Vaccination and Smallpox as observed in the Cases admitted to the City of Glasgow Smallpox Hospital, Belvidere, between 10th April, 1900, and 30th June, 1901*, by Dr R. S. Thomson and Dr Fullarton—a paper read before the Royal Philosophical Society of Glasgow on 2nd April, 1902—and from supplementary statistics relating to the same epidemic kindly sent to me by Dr Brownlee, Physician Superintendent of the Belvidere Hospital. The tables were prepared, and the coefficients of correlation with their probable errors calculated in the same manner as in the earlier paper. I may repeat that I use "mild" as equal to "discrete," and "severe" as equal to "confluent" and "haemorrhagic." The few cases (only $2\frac{1}{2}$ per cent.) of "doubtful" vaccination have been excluded from my tables. The following are my tables for the Glasgow epidemic of 1900–1.

TABLE I.

	Recoveries	Deaths	Totals
Vaccinated...	1493	150	1643
Unvaccinated	59	63	122
Totals	1552	213	1765

$r = \cdot 6294 \pm \cdot 0296.$

TABLE II.

	Mild	Severe	Totals
Vaccinated...	1361	282	1643
Unvaccinated	53	69	122
Totals	1414	351	1765

$r = \cdot 5162 \pm \cdot 0322$.

TABLE III.

Scars	Mild	Severe	Totals
Foveated ...	573	82	655
Unfoveated...	788	200	988
Totals	1361	282	1643

$r = \cdot 1929 \pm \cdot 0374$.

TABLE IV.

Area of Scar	Mild	Severe	Totals
Over half square inch	805	113	918
Half square inch and under	556	169	725
Totals	1361	282	1643

$r = \cdot 2646 \pm \cdot 0301$.

TABLE V.

Number of Scars	Mild	Severe	Totals
Two and upwards	652	105	757
One	709	177	886
Totals	1361	282	1643

$r = \cdot 1511 \pm \cdot 0306$.

TABLE VI.

Vaccinated Cases.

	Recoveries	Deaths	Totals
Mild	1336	25	1361
Severe... ...	157	125	282
Totals	1493	150	1643

$r = \cdot 8603 \pm \cdot 0150$.

The statistics also enable me to find the coefficient of correlation between age and severity of attack.

TABLE VII.

Vaccinated Cases.

Years	Mild	Severe	Totals
0 to 20... ...	244	14	258
20 and over...	1117	268	1385
Totals	1361	282	1643

$r = \cdot 3693 \pm \cdot 0386$.

If the division be made at 25 years instead of 20, $r = \cdot 3297 \pm \cdot 0314$; if at 35 years, $r = \cdot 3180 \pm \cdot 0293$. Calculating this coefficient for the Glasgow epidemic of 1892–95, I find $r = \cdot 3366 \pm \cdot 0715$, for a division at 20 years; I am unable to calculate it for a division at 25 or 35 years, as there are no divisions at these periods in the statistics of the earlier epidemic.

(2) For the sake of comparison, the coefficients for the two Glasgow epidemics are collected in the table below.

TABLE VIII.

Coefficient of correlation between	Epidemic of 1900–1	Epidemic of 1892–5
Vaccination and strength of resistance	$\cdot 6294 \pm \cdot 0296$	$\cdot 7783 \pm \cdot 0365$
Vaccination and severity of attack	$\cdot 5162 \pm \cdot 0322$	$\cdot 9123 \pm \cdot 0181$
Foveation of scar and severity of attack	$\cdot 1929 \pm \cdot 0374$	$\cdot 3951 \pm \cdot 0594$
Area of scar and severity of attack	$\cdot 2646 \pm \cdot 0301$	$\cdot 3520 \pm \cdot 0584$
Number of scars and severity of attack	$\cdot 1511 \pm \cdot 0306$	$\cdot 2323 \pm \cdot 0616$
Strength of resistance and severity of attack ...	$\cdot 8603 \pm \cdot 0150$	—
Age (division at 20 years) and severity of attack	$\cdot 3693 \pm \cdot 0386$	$\cdot 3366 \pm \cdot 0715$
Age („ 25 „) „ „ „	$\cdot 3297 \pm \cdot 0314$	—
Age („ 35 „) „ „ „	$\cdot 3180 \pm \cdot 0293$	—

It will be observed that in the recent epidemic the coefficient of correlation between vaccination and strength of resistance, while less than in 1892–95, was very much the same as in previous epidemics in other towns*. On the other hand, the correlation between vaccination and degree of severity of attack is very much less than in 1892–95, which points to a marked difference in the character of the two epidemics, the earlier being the milder. Also the correlation between degree of severity and (1) foveation, (2) area, and (3) number, of scars is less even than in 1892–95; but I understand from Dr Brownlee that in concluding whether a scar is good or bad, clinically, he would take into consideration its area,

* *Biometrika*, Vol. I. No. 3, p. 380.

the area of foveation, and the amount of depression and "puckering." The fact is also clearly brought out that adults are considerably more liable to the severe forms of the disease than the young.

(3) The next part of my inquiry presents some features which, I think, are novel, and is based on figures relating to the Glasgow epidemic of 1900–1, supplied by Dr Brownlee. He grouped the male patients according to their occupations, adopting the grouping of the Registrar-General's Reports with three modifications: (1) Professions, Clerks, etc. are all grouped together, as the number of patients other than clerks was very small; e.g. only four professional men were admitted to Belvidere: (2) Railway men are shown separately from Transport Service: (3) Shopkeepers include all kinds, instead of the limited number adopted in the Registrar-General's Reports. Cases of doubtful vaccination (25) are again excluded.

The following table is prepared from Dr Brownlee's figures:

TABLE IX.

Smallpox Cases admitted into Belvidere Hospital 1900–1. (*Males.*)

Occupations	Vaccinated			Unvaccinated
	Recoveries	Deaths	Totals	
Miners...	25	3	28	2
Labourers	167	25	192	9
Metal Workers	159	14	173	0
Shopkeepers	70	10	80	5
Railway men	13	2	15	1
Transport Service ...	85	11	96	3
Other Trades	46	10	56	3
Spirit Salesmen... ...	11	3	14	0
Professions, Clerks, etc.	65	6	71	4
Building Trades ...	74	9	83	1
Textiles	22	1	23	0
Unoccupied	10	1	11	1
Totals	747	95	842	29

From the statistics of Table IX. I wanted to ascertain the correlation between the social status of the patients and (1) the presence or absence of the scar, and (2) in the case of the vaccinated their power to resist the disease. For this purpose it was necessary to divide my material into two classes having a higher and a lower status. No doubt considerable diversity of opinion may exist as to the components which should fall into these two classes. I consulted from this standpoint the death-rates of various classes as given by the Registrar-General.

My chief difficulty arose from the group of metal workers who form by far the largest single occupation group after the labourers, and yet exhibit not a single unvaccinated person. It is clear that such a class according as it is included in the higher or lower status group may completely change the correlation between status and vaccination. If the metal workers included filecutters and lead-workers, they would have a high death-rate and we ought to put them in the group of lower status. On the other hand in some districts the metal workers belong to the most trained and the best paid class of craftsmen, men who are likely to be well nourished and with fairly healthy homes. I accordingly wrote to Dr Brownlee inquiring as to the status of the Glasgow metal workers and as to the absence of unvaccinated cases among them. His reply is as follows:

"No filecutters or leadworkers were included in the metal workers. The latter "were in great majority made up of shipbuilding and forge employees, these "being the main metal industries in Glasgow. A very few brassworkers were "included. The status of these metal workers as we received them was not so high "as that of either the men who work on the Railways or in the Building Trades, "and I think that they should be included in your second class. The absence of "unvaccinated cases is to a certain extent explainable. Most of these workers were "from the neighbouring forges where on account of the contiguity of the hospital "the employers have for a long time exercised a certain amount of supervision of "the vaccination of those employed in the works. Smallpox has several times "invaded these works since 1890, so that I think unvaccinated persons must "be very few indeed."

It must be at once confessed that this more or less enforced vaccination of the large group of metal workers is a very disturbing factor in the consideration of the relation between status and vaccination. Adopting Dr Brownlee's view I first put the metal workers in the class of lower status.

As a first grouping I took for those of higher status: Professions, Clerks, etc., and Shopkeepers. I obtained:

TABLE X.

First Grouping.

Class	Vaccinated	Unvaccinated	Totals
Lower	691	20	711
Higher ...	151	9	160
Totals	842	29	871

$r = \cdot 1862 \pm \cdot 0737$.

I now included in the class of higher status, Miners, Railway men and Building Trades as well as Professions, Clerks and Shopkeepers.

TABLE XI.

Second Grouping.

Class	Vaccinated	Unvaccinated	Totals
Lower	565	16	581
Higher ...	277	13	290
Totals	842	29	871

$r = \cdot 1362 \pm \cdot 0695$.

There is thus for both cases a quite sensible if not very large correlation between status and vaccination,—*cases of unvaccinated persons occur more frequently in the classes of higher than of lower status.*

Putting on one side Dr Brownlee's views on the status of the metal workers I now formed two tables in which the metal workers were included in the group of higher status, adding them first to Professions, Clerks, etc. and Shopkeepers, as a third grouping, and finally to all these together with Miners, Railway men and Building Trades as a fourth grouping. I found:

TABLE XII.

Third Grouping.

Class	Vaccinated	Unvaccinated	Totals
Lower	518	20	538
Higher ...	324	9	333
Totals	842	29	871

$r = -\cdot 0872 \pm \cdot 0713$.

TABLE XIII.

Fourth Grouping.

Class	Vaccinated	Unvaccinated	Totals
Lower	392	16	408
Higher ...	450	13	463
Totals	842	29	871

$r = -\cdot 0939 \pm \cdot 0691$.

The correlation has clearly swung round, and there is now a very slight correlation, hardly sensible considering the probable error, between higher status and vaccination. In view, however, of Dr Brownlee's opinion as to the character

of the metal workers, I do not think this classification is legitimate. Considering also that vaccination appears to be more or less compulsory among a considerable section of them I determined to omit them altogether, and then obtained the following table:

TABLE XIV.

First Grouping, Metal Workers excluded.

Class	Vaccinated	Unvaccinated	Totals
Lower	518	20	538
Higher ...	151	9	160
Totals	669	29	698

$r = \cdot 1167 \pm \cdot 0763$.

Comparing this with Tables X. and XII. we see that there is a sensible but small correlation between higher status and unvaccinated condition.

Now I lay no particular stress on these results because the material is far too sparse*, but I believe that the above statistics are the only ones hitherto dealt with with a view to determining whether the classes of higher status—presumably the better fed and healthier classes of the community—are or are not more frequently vaccinated than the lower, presumably the less nourished and less healthy classes. No dogmatic conclusion can be drawn from these data, but they exhibit no evidence at all for the unvaccinated class being of lower status than the vaccinated class; on the contrary, there is slight evidence to show that the unvaccinated in Glasgow occur rather more frequently in the classes of higher status.

(4) I turn now to the question whether among the vaccinated there exists a correlation between status and severity of the disease. I obtained the following tables:

TABLE XV.

First Grouping.

Class	Deaths	Recoveries	Totals
Lower	79	612	691
Higher ...	16	135	151
Totals	95	747	842

$r = \cdot 0249 \pm \cdot 0566$.

* The fewness of unvaccinated cases possibly arises from the fact that during the 1892–95 epidemic in Glasgow vaccination was performed on a large scale amongst all classes, so that the epidemic of 1900–1 found the great majority of the population vaccinated.

TABLE XVI.

Second Grouping.

Class	Deaths	Recoveries	Totals
Lower	65	500	565
Higher ...	30	247	277
Totals	95	747	842

$r = \cdot 0216 \pm \cdot 0500$.

That is to say there is a positive but practically insensible correlation between status and power to resist the disease.

I now included the metal workers in the higher status and found with the former groupings:

TABLE XVII.

Third Grouping.

Class	Deaths	Recoveries	Totals
Lower	65	453	518
Higher ...	30	294	324
Totals	95	747	842

$r = \cdot 1086 \pm \cdot 0490$.

TABLE XVIII.

Fourth Grouping.

Class	Deaths	Recoveries	Totals
Lower	51	341	392
Higher ...	44	406	450
Totals	95	747	842

$r = \cdot 1052 \pm \cdot 0478$.

Thus the correlation between better status and recovery is slightly increased. Leaving the metal workers as before out of consideration, I find:

TABLE XIX.

First Grouping without Metal Workers.

Class	Deaths	Recoveries	Totals
Lower	65	453	518
Higher ...	16	135	151
Totals	81	588	669

$r = \cdot 0579 \pm \cdot 0594$.

This is probably the most satisfactory result as about the mean of the previous tables. We accordingly conclude that there is a very slight relation between status and recovery from attack.

The way in which the metal workers increase this correlation when they are placed in the better class is remarkable; they have in Glasgow a comparatively small mortality from smallpox, yet we find from the *55th Annual Report for England of the Registrar-General—Medical Supplement* that the general mortality among metal workers is higher at all ages, especially after 35, than that of occupied males. The figures are as follows, if the standard rate of mortality among occupied males at each age be taken as 100:

Ages	15—	20—	25—	35—	45—	55—	65 and up.
Metal Workers	105	106	103	111	122	129	128

It would be interesting to compare the special classes grouped as metal workers in Glasgow, with those embraced under the same heading in the Registrar-General's Reports.

(5) *Conclusions.*

(i) The statistical constants for vaccination and smallpox differ sensibly for the same place with two different epidemics, i.e. epidemics seem to be differentiated in character.

(ii) The statistics of Glasgow do not indicate that those of lower status —and therefore probably worse nourished and housed—provide the bulk of non-vaccinated cases. On the contrary there seems to be a slight tendency for the non-vaccinated to be of higher status.

(iii) There is a slight although scarcely sensible correlation between status and power to resist a smallpox attack.

The Glasgow statistics do not go very far, but as far as they go they do not justify the statement: That the apparent protection of vaccination is due to the unvaccinated belonging to classes of lower status which have a far smaller power of resistance to the disease than the better nourished classes of a higher status in which the members are more generally vaccinated.

I have not dealt with the statistics as to female patients as the great bulk of them are classified merely as "Housewives," which throws no light on their social position.

In concluding this paper I venture to express the hope that the statistics of the recent London epidemic may soon be issued and in a form which will admit of due consideration of the problems referred to in this paper. Their magnitude gives them extreme value and their publication is no doubt anxiously awaited by a wide circle of scientific inquirers at home and abroad.

NOTE.

From the "Times" of 20 November 1902, it appears that the Medical Officer of Health for Islington has prepared his final report on the 1901–2 epidemic so far as his own district was concerned, but I have been unable to obtain a copy of it. The paragraph in the "Times", however, enables me to form the following table:

TABLE XX.

Smallpox in Islington, 27 *August* 1901—29 *August* 1902.

	Recoveries	Deaths	Totals
Vaccinated ...	207	29	236
Unvaccinated...	35	30	65
Totals	242	59	301

$r = \cdot 5744 \pm \cdot 0560.$

This value of r is in extremely close agreement with that obtained for 1017 cases which were "completed" in London in 1901 *, viz. $r = \cdot 5779 \pm \cdot 0311$. To show that the closeness of the agreement holds throughout the investigation, I give the equations from which r was calculated in both cases:

For the above 301 cases:

$$\cdot 070608\, r^6 + \cdot 002326\, r^5 + \cdot 151410\, r^4 + \cdot 017030\, r^3 + \cdot 336384\, r^2 + r = \cdot 707743.$$

For the 1017 cases:

$$\cdot 071607\, r^6 + \cdot 001780\, r^5 + \cdot 149636\, r^4 + \cdot 0198442\, r^3 + \cdot 326092\, r^2 + r = \cdot 710100.$$

* *Biometrika*, Vol. I. No. 3, p. 379.

COOPERATIVE INVESTIGATIONS ON PLANTS.

II. VARIATION AND CORRELATION IN LESSER CELANDINE FROM DIVERS LOCALITIES.

(1) In view of the data for *F. ranunculoides* published by Dr F. Ludwig* and Prof. MacLeod† and the statistical constants determined for them‡ it seemed desirable to obtain rather more statistical material and a more comprehensive series of constants for the purposes of a comparative study. There were two points to be considered, namely: (i) the influence of locality and (ii) the influence of the time during the flowering season at which the flowers were gathered. Unfortunately we were in the present season in no case able to obtain from our collectors two series from the same locality with a month's interval between the gatherings. All the collecting except in one case had to be done during the brief Easter vacation of our workers, and this did not admit of a double gathering in the same locality at a suitable interval. The one exception is that of the Bordighera collection. Mr Francis Galton most kindly offered, as he was wintering on the Mediterranean, to provide a double series of Lesser Celandine flowers. The first series was gathered about February 19th and at once dispatched to England. On arrival it was found that practically every sepal, every petal, and nearly every stamen had fallen from the flower-heads. This lost us our earlier series, but we learnt a most valuable lesson, namely: that transit of any kind, even by hand, will cost the flower if nearly full-blown one or more sepals, petals or stamens. Our plans had therefore to be changed; each celandine flower was now gathered *as a bud* and wrapped up in a small piece of tissue paper. This involved a great increase in the labour of gathering and a much greater one in that of counting, a good deal of which had to be done under a lens, but we were thus certain of preserving all the parts of the flower intact. Mr Francis Galton suggested and carried out this arrangement in a second series gathered between March 4th and 7th which reached England safely, but three or four weeks later than this there were no flowering celandines to be obtained in Bordighera.

Mr Galton's plan was carried out in the further collections made in Guernsey, Dorset and Surrey, the collectors gathering the buds, and wrapping them up

* *Biometrika*, Vol. I. pp. 11—20. † *Ibid.*, pp. 125—128.
‡ *Ibid.*, pp. 316—319.

in situ in a small piece of tissue paper. The buds dried and were preserved for weeks, to be counted at leisure. Even if a petal or sepal fell off it was preserved in the paper wrapper. We believe it is largely due to this method of gathering that our results show such a totally different distribution of sepals and petals to those of other observers.

Take the case of sepals. In the Guernsey series we found only three individuals with less than three sepals. On examination under the microscope in one of these cases an abortive sepal was found, in another the sepal had clearly once been attached, but in the third case the head was unfortunately lost before microscopic examination. We think that we may safely affirm that no true case of less than three sepals was found in the Guernsey plants. In the Dorsetshire gathering no cases of less than three sepals occurred. In the Surrey gathering there were in the material six cases of less than three sepals, and in five out of these six cases the rule of gathering buds only had been disregarded by a young collector*; and the sixth was an aborted flower in which the stamens were not properly developed and there were no pistils at all! In the Bordighera celandines there are only two cases of less than three sepals. These have not been excluded from the calculations, because we had not when counting them learnt from the Dorset and Guernsey series to regard all cases of less than three sepals with grave suspicion and examine such cases under the microscope. But if in 624 cases only two such individuals occurred, it seems extremely probable that even these were cases of sepals knocked off or aborted. In a total of 2149 heads, there were 11 cases of less than three sepals, two of these were aborted flowers, five of them were old flowers with parts loose, one had once had a sepal which had been lost, and three were not closely examined. The experience we have had leads us to believe that each flower ought to be gathered as a bud and at once wrapped up. It seems to us that all we can admit is a *possibility* of three heads with less than three sepals in 2149 cases, while Dr Ludwig's 3000 from Greiz show no less than 60 definite cases†. We feel fairly confident that had the Greiz flowers been collected as buds and possibly gathered and counted by adults instead of school children they would not have differed so widely from our material in this respect. As far as our experience reaches, we question the existence of any normal flower with less than three sepals. This view may be modified when further material from central Europe gathered in bud and, if necessary, microscopically examined, is available. We hope next season that this problem of the sepal may be directly investigated.

(2) *Material of the present investigation.*

(A) 624 heads gathered in a vineyard at Bordighera, Italy, between March 4th and 7th. Due to Mr Francis Galton. The sepals, petals, stamens and pistils were counted by Mr N. Blanchard. The tables were prepared and the statistical

* The calculator's notes are: No. 51, "only one sepal and this hanging loose"; No. 52, "old flower with petals detached"; No. 53, "Pistils loose"; No. 64, "very old flower, all parts loose"; No. 69, "very old flower." No remarks as to age of this kind occur in other cases.

† See *Biometrika*, Vol. I. pp. 13—15.

constants calculated by Miss Alice Lee. These belonged to a variety of Lesser Celandine classed by the French botanists as *Ficaria calthaefolia* and considered by them as distinct from *F. ranunculoides*. Table I., however, exhibits less differentiation in the floral parts of *F. calthaefolia* and *F. ranunculoides* than can occur between two local races of the latter. The *Index Kewensis* gives the former name as a synonym of the latter, but there appears to be a more sensible differentiation of the leaf.

(B) 520 heads gathered on roadside banks at St Peter Port, Guernsey, between April 25th and 30th. Due to Miss Caroline Herford. The characters were counted, the tables formed and the statistical constants determined by Miss Mary Beeton.

(C) 500 heads gathered in two lanes and a meadow near Thursley, on the north side of Hind Head, Surrey. The material was gathered between 12th and 17th of April by a number of collectors under the supervision of K. Pearson. The characters were counted by Mr N. Blanchard and the tables and statistical constants are due to Dr Lee.

(D) 505 heads gathered in a lane at Studland, Dorsetshire, on April 7 by Mr N. L. Blanchard. The counting is due to Miss Edna Lea-Smith and to Mr Blanchard. The statistical tables and constants are again due to Dr Lee.

In reducing the material Sheppard's corrections for the moments were not used, partly because we are dealing with variation by units, and partly because it is clear in the case of the sepals, and probably true for the petals, that the frequency distribution has not high contact at the low end of the range. Otherwise the calculation of means, standard deviations, coefficients of correlation, and of variation proceeded in the usual manner.

(3) Table I. gives a summary of all the statistical constants* for the four new series and places alongside them those already found for other localities. Now this table shows at once the remarkable differences, which period in season and environment can have on mean, variability and correlation. Not one of these quantities has the least approximation to constancy for all local races of *Ficaria*, nor for the same race at two parts of the same reason. The early Belgian celandines, judging from the constants for stamens and pistils, which are all we have, are practically identical with those from Italy, while the later ones diverge widely and perhaps may be considered nearer to the Swiss Trogen series than to any other. A brief study of this table will at once convince the reader of two fundamental points:

(*a*) Local races in plants cannot be defined or distinguished by the existence of differences many times the value of their probable errors between their means, variations or correlations.

(*b*) The influences of environment and season are for plants of supreme importance and very widely or indeed entirely screen any differences due to local race.

* These are tabulated to four places of decimals—not because such are exact or necessary for present purposes, but because for future investigations when we come to consider the evolutionary history of local races, it will be needful to have the constants to this number of places in order to calculate the partial regression coefficients true for each series to one or two places of decimals.

TABLE I. *Statistical Constants of Lesser Celandine from different localities.*

Locality		Italy	Guernsey	England		Belgium		Germany		Switzerland		Europe
		Bordighera	St Peter Port	Surrey	Dorset	Early	Late	Greiz	Gera *	Trogen	Gais	Means
Year		1902	1902	1902	1902	1899	1899	1900–1901	?	1900	1901	1899–1902
Date		March 4–7	April 25–30	April 12–17	April 7–9	Feb. 27—March 17	April 17–23	?	?	?	?	—
Number		624	520	500	505	268	373	4000	1000	285	1000	—
Means	Sepals	3·0625 ± ·0071	3·0096 ± ·0029	3·0100 ± ·0030	3·0297 ± ·0054	—	—	3·650 ± ·0171	3·309 ± ·0137	3·386 ± ·0219	4·538 ‡ ± ·0131	3·374
	Petals	8·1186 ± ·0117	8·1327 ± ·0137	8·0680 ± ·0139	8·7149 ± ·0304	—	—	8·351 ± ·0260	8·225 ± ·0237	8·144 ± ·0306	9·722 ‡ ± ·0273	8·435
	Stamens... ...	26·7179 ± ·1018	27·9558 ± ·1419	32·9380 ± ·1721	35·5881 ± ·1803	26·7313 ± ·1550	17·8633 ± ·1152	—	—	20·3682 ± ·1528	23·8250‖ ± ·2177	26·498
	Pistils	17·7051 ± ·1068	23·9115 ± ·1877	24·5360 ± ·1931	28·3287 ± ·2242	17·4478 ± ·1604	12·1475 ± ·1183	—	—	13·2635 ± ·1223	18·1125‖ ± ·3234	19·432
Standard Deviations	Sepals	·2612 ± ·0050	·0976 ± ·0020	·0995 ± ·0021	·1811 ± ·0038	—	—	·8037 ± ·0121	·6406 ± ·0097	·5487 ± ·0155	·6158‡ ± ·0093	·4060
	Petals	·4334 ± ·0083	·4638 ± ·0097	·4598 ± ·0098	1·0118 ± ·0215	—	—	1·2189 ± ·0184	1·1113 ± ·0168	·7663 ± ·0216	1·2819‡ ± ·0193	·8434
	Stamens ...	3·7720 ± ·0720	4·7978 ± ·1003	5·7039 ± ·1217	6·0058 ± ·1275	3·7609 ± ·1096	3·2984 ± ·0815	—	—	3·8234 ± ·1080	2·8872‖ ± ·1540	4·2562
	Pistils	3·9568 ± ·0755	6·3470 ± ·1327	6·4001 ± ·1365	7·4713 ± ·1586	3·8943 ± ·1135	3·3878 ± ·0837	—	—	3·0606 ± ·0865	4·2885‖ ± ·2288	4·8508
Coefficients of Variation	Sepals ...	8·53	3·24	3·31	5·98	—	—	22·02	19·36	16·20	13·57	11·53
	Petals ...	5·34	5·70	5·70	11·61	—	—	14·60	13·51	9·41	13·19	9·88
	Stamens...	14·12	17·16	17·32	16·88	14·07	18·46	—	—	18·77	12·12	16·11
	Pistils ...	22·35	26·54	26·08	26·37	22·32	27·89	—	—	23·08	23·68	24·79
Correlations	Sepals & Petals...	·1044 ± ·0267	·0993 ± ·0293	·2912 ± ·0276	·2300 ± ·0284	—	—	·2451 ± ·0200	·1928 ± ·0205	[−·1821?]† ± ·0386	·2237 ‡ ± ·0203	·1981
	Sepals & Stamens	·0228 ± ·0270	·0584 ± ·0295	·0222 ± ·0300	·2225 ± ·0285	—	—	—	—	—	—	·0815
	Sepals & Pistils...	·0706 ± ·0269	·0324 ± ·0295	·1078 ± ·0297	·2490 ± ·0282	—	—	—	—	—	—	·1150
	Petals & Stamens	·2243 ± ·0256	·2939 ± ·0264	·2690 ± ·0280	·3825 ± ·0256	—	—	—	—	—	—	·2924
Petals and Pistils...		·1961 ± ·0260	·2222 ± ·0281	·3508 ± ·0265	·3527 ± ·0263	—	—	—	—	—	—	·2805
Stamens and Pistils		·4387 ± ·0218	·5206 ± ·0216	·6601 ± ·0170	·6682 ± ·0166	·5065 ± ·0306	·7489 ± ·0153	—	—	·5328 ± ·0290	·3913‖ ± ·0639	·5584

* A second series from Gera while less variable gives sensibly the same means and correlation. A third series, more or less intermediate in variation, gives an insensible negative correlation: see *Biometrika*, Vol. I. p. 317.

† This result is very discordant and is not included in the European mean: see *Biometrika*, Vol. I. p. 317.

‡ A second short series for Gais, with somewhat different means and variabilities, is fairly accordant in correlation.

‖ 80 heads only from 1900.

These results form a further proof, were any needful, that race cannot be defined by correlation*. Plants are, of course, more subject to environment than animal life, and the means and variabilities change with change of environment and of season very readily. Man, perhaps, of all animal life seems least affected by environment, but a table like the above showing how different the mean, variability and correlation are when there is slight change in locality or season, ought to render us cautious of cataloguing even local races of man by the means alone of small craniological series. Nurture and environment during growth may sensibly influence even human mean characters. The biometrician, far from being discouraged by the fact that the simpler biometric quantities—means, variabilities, correlations—are not constant for local races, ought rather to recognise that if natural selection be a reality these are the very quantities which he would expect to change from one local race to another, and from one environment to a second. Indeed it is exactly these differences which form the foundation upon which we hope to build up in the future the evolutionary history of local races; they are the material from which we must determine not only the characters which in each case have been selected, but also the absolute constants which define the species itself†.

(4) Comparing the first four columns—the new data—with the last four columns—the old data for central Europe—one is at once struck with the high degree of variability in sepals and petals of the latter as compared with the former. The sepals are more than four times, the petals are nearly twice, as variable in the second group, i.e. the mean values are ·6544 and 1·0946 as against ·1598 and ·5944 respectively. Now we see in the case of the Dorset group that the petal variability can rise to double the value it had in Bordighera, Guernsey or Surrey. But an examination of the tables below shows that much of the difference arises from the absence of individuals with less than 3 sepals or 7 petals in the first group, and as our data are drawn from very diverse districts we must await with interest the result of *bud* countings from Germany.

TABLE II.

Frequency of Sepals per 1000.

Number of Sepals	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Bordighera	—	—	3·2	932·7	62·5	1·6	—	—
St Peter Port ...	—	—	—	990·4	9·6	—	—	—
Surrey...	—	—	—	990·0	10·0	—	—	—
Dorset...	—	—	—	972·3	25·7	2·0	—	—
Greiz	4·2	1·8	9·0	546·8	251·7	174·2	11·5	0·8
Gera	3·0	0·0	21·0	733·0	152·0	86·0	5·0	—
Trogen	—	—	—	680·7	259·7	52·6	7·0	—
Gais	—	—	2·0	91·0	283·0	616·0	7·0	1·0

* *Phil. Trans.* Vol. 187, A, pp. 266, 280.

† "Mathematical Contributions to the Theory of Evolution. XI. On the Influence of Natural Selection on Variation and Correlation." *Phil. Trans.* Vol. 200, A, pp. 18—21.

TABLE III.

Frequency of Petals per 1000.

Number of Petals	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
Bordighera ...	—	—	—	1·6	14·4	868·6	96·2	17·6	1·6	—	—	—	—
St Peter Port	—	—	—	—	13·5	869·2	96·1	13·5	7·7	—	—	—	—
Surrey ...	—	—	—	4·0	40·0	858·0	84·0	10·0	4·0	—	—	—	—
Dorset ...	—	—	—	—	7·9	560·4	231·7	124·7	63·4	7·9	4·0	—	—
Greiz	0·2	0·8	6·8	32·7	148·0	473·0	180·0	99·0	35·3	17·2	6·0	0·5	0·5
Gera	—	—	16·0	31·0	112·0	563·0	175·0	62·0	24·0	11·0	5·0	0·0	1·0
Trogen ...	—	—	3·5	14·0	119·3	642·1	150·9	59·7	10·5	—	—	—	—
Gais	—	—	—	2·0	34·0	139·0	265·0	296·0	175·0	69·0	18·0	2·0	—

But making all allowance for this the German and Swiss series show a marked increase in the number of both sepals and petals over our series, culminating in Gais with its modes of 5 sepals and 10 petals. Our Dorsetshire series indicates a substantial advance in the same direction as far as the petals are concerned, the Bordighera tends somewhat in the same direction for the petals. It remains as a task for next spring to determine whether the Guernsey and Gais series cannot be linked together by a continuous system of intermediate series, if only the flowers be gathered at a sufficiently wide range of places and at different seasons in those places.

(5) Turning to the correlations which involve sepals one is hardly surprised at their irregularity, for their values depend in our new data on the distribution of a very few individuals with more than four sepals. These for Bordighera are less than 7 per cent., for Dorset less than 3 per cent., and for Guernsey and Surrey not more than 1 per cent.! Hence one or two irregular or anomalous individuals cause the correlation to alter in a very remarkable manner. A much safer set of correlations are those between petals, stamens and pistils. Unfortunately the counting of stamens and pistils is a much harder task, and for a series of 500 bud-gatherings requires a week or two of very laborious work.

Looking at our new data, which are arranged according to number of stamens, we see several important and almost uniform results:

(*a*) If the different series be arranged in ascending order of stamens, this will also be an ascending order for the number of pistils.

With one exception—Surrey—this is also an ascending order for the number of petals.

(*b*) The ascending order for stamens or pistils is also an ascending order for the variability of both stamens and pistils.

With the same exception—Surrey—it is an ascending order for variability of petals.

(*c*) The ascending order for means or variabilities of stamens or pistils is an ascending order for the three correlations between stamens, pistils and petals.

The slight exception to this rule—Surrey correlation of petals and stamens—is well within the probable error of the determinations.

Summing up these results in a single statement, we conclude from the data that were gathered and counted on one uniform plan that:

The local races of the lesser celandine which have flowers with more numerous parts, have those parts more variable and more highly correlated.

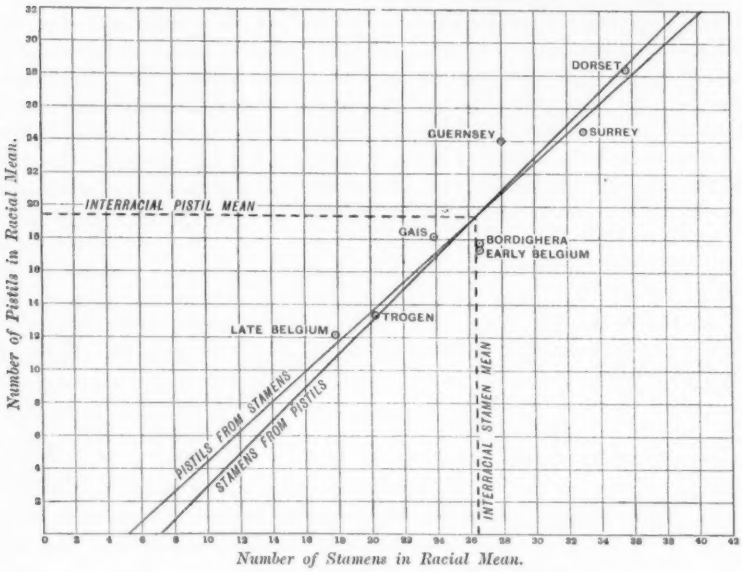
Let us compare these results with the data obtained by other observers. Germany contributes nothing, for stamens and pistils have not yet been counted. The Swiss data, if individually a little anomalous, form, if clubbed together, a striking confirmation of our argument,—they have fewer pistils and stamens than our four series, the variations are less, and the correlations for stamens and pistils are low—Gais has the lowest result of all, and Trogen only about the Guernsey value. Unfortunately we can only compare one correlation and this has been worked out for only small numbers, 285 in one case and 80 in the other. With regard to Belgium the comparative results are not so satisfactory. The early Belgium flowers have about the same number of stamens as the Italian and fewer pistils, the variability of both stamens and pistils is less. The late Belgium flowers have the least number of stamens and pistils of any series, their variability in both stamens and pistils is less than all four of our series, and less than the mean of the two Swiss series. Here again we have unfortunately only one correlation available for comparison, that of stamens and pistils. It is for the early flowers somewhat greater than that for the Italian series, when it ought to be about equal to it, while for the late flowers, it is the highest of all when it should be the least from the above rule. But just in this material a new factor comes into play, which, if not the source of the result, deserves to be taken into account. The late flowers are said to be from the *same* plants as the early flowers. But pistils and stamens cannot be counted unless the flower be gathered. Hence the plants from which the later flowers are gathered may be heterogeneous in character, namely plants naturally flowering late, and plants which have been more or less injured by one or more flowers being removed and which have reflowered in a more or less exhausted condition. Such heterogeneity would tend to increase correlation. Clearly in this as in so many cases the conditions are very complex, and we cannot without further investigation club together flowers from plants (*a*) which have had earlier flowers gathered, (*b*) which naturally flower late, with the late flowers of plants which have been flowering throughout the season undisturbed, and term the group 'late flowers' without danger of introducing heterogeneity. Till therefore the whole subject of the correlation of late and early flowers has been gone into carefully with ample data, we may take, we hold, the above general principle—that increase in the number of parts denotes increase in the variability and correlation of those parts—as a working hypothesis to be demonstrated or modified by further researches. We, of course, do not extend the principle beyond the species for which the data are deduced, even as a "working hypothesis," and we state it only for petals, stamens and pistils.

(6) It may be useful to examine the interracial result from the correlation standpoint, although at present we have rather meagre data. We have eight local series; considering them of equal weight, although the Belgian and Swiss series are very inferior in number to ours, we find:

Mean of local means for stamens	= 26·498 ± 1·321
Mean of local means for pistils	= 19·432 ± 1·265
Variability of local means for stamens ...	= 5·5409 ± ·9343
Variability of local means for pistils ...	= 5·3037 ± ·8943
Correlation of local means for stamens and pistils	= ·9514 ± ·0235.

The probable errors are very high because we have only eight pairs. It is therefore difficult, perhaps, to make a definite statement that the interracial is greater than the average intraracial variability, but it appears like it; even the relative

Lesser Celandine. Interracial Regression Lines. Stamens and Pistils.



order of variability of pistils and stamens may be changed when we pass from intraracial to interracial constants. On the other hand the correlation is so high * that we can make a very definite statement about it, i.e.:

The interracial correlation of the mean numbers of stamens and pistils is very much greater than the mean intraracial correlation between stamens and pistils, being to the latter nearly in the ratio of 12 *to* 7.

* And therefore its probable error even with only eight pairs, so small.

These conclusions show us how extremely cautious we must be in extending conclusions deduced by intraracial calculations to interracial relations. Intraracial and interracial variabilities and correlations may be of a totally different order. By the usual theory of regression we have the following formulæ to find the probable number of stamens in a race given the mean number of pistils, and the probable number of pistils given the mean number of stamens:

$$\text{Probable mean number of stamens in a local race} = \cdot 9939 \times (\text{observed mean number of pistils}) + 7 \cdot 185.$$

$$\text{Probable mean number of pistils in a local race} = \cdot 9106 \times (\text{observed mean number of stamens}) - 4 \cdot 697.$$

Comparing with actual values we have:

	Stamens from Pistils		Pistils from Stamens	
	Observed	*Calculated*	*Observed*	*Calculated*
Bordighera	26·7	24·9	17·7	19·6
St Peter Port	28·0	31·0	23·9	20·8
Surrey	32·9	31·7	24·5	25·3
Dorset	35·6	35·3	28·3	27·7
Early Belgian	26·7	24·5	17·4	19·6
Late Belgian	17·9	19·3	12·1	11·6
Trogen	20·4	20·4	13·3	13·8
Gais	23·8	25·2	18·1	17·0

The maximum error in the number of stamens is 3·0 and the mean error 1·3; for the number of pistils it is 3·1 and the mean error again 1·3. Thus our formulæ will give the number of stamens from pistils, or pistils from stamens in any race with an average error of about one pistil or stamen, and a maximum of about three. They are thus close enough for most practical purposes. The diagram gives the lines to read stamens from pistils and pistils from stamens.

(7) There are a number of other points which are suggested by our table, but it seems better to defer their discussion until more ample data are forthcoming. There is hardly any district from which at present a series of 500—600 celandine heads would not be of value. But these series ought to be gathered on a uniform plan, i.e. gathered as buds, each bud being wrapped in a separate soft paper cover. Further since the labour of counting is very great, those who will count as well as gather are the more valuable helpers. In counting great care must be taken to separate the individual pistils, and all heads with less than three sepals ought to be carefully examined with a microscope or powerful lens. If possible two series should be taken from each locality, one early and one late in the season, but the flowers should not be taken off the same plants, but off plants having substantially the same environment. A record of date, environment and

locality should accompany each series. The most important localities at present seem to be, Mid-France, South and Mid-Italy, Mid and North England, Switzerland and South Germany in as many districts as possible, and any northern European stations, where workers may be available. It would be of great value to have one species of plant thoroughly worked out from the biometric standpoint over a wide area. This can only be done by cooperative labour.

CORRELATION TABLES. LESSER CELANDINE.

A. SEPALS AND PETALS.

I. *Bordighera.*

Sepals.

Petals.	*2*	*3*	*4*	*5*	Totals
6	—	1	—	—	1
7	—	3	6	—	9
8	2	521	19	—	542
9	—	46	14	—	60
10	—	10	—	1	11
11	—	1	—	—	1
Totals	2	582	39	1	624

II. *Guernsey.*

Sepals.

Petals.	*3*	*4*	Totals
7	6	1	7
8	450	2	452
9	49	1	50
10	7	—	7
11	3	1	4
Totals	515	5	520

III. *Surrey.*

Sepals.

Petals.	*3*	*4*	Totals
6	2	—	2
7	20	—	20
8	429	—	429
9	38	4	42
10	5	—	5
11	1	1	2
Totals	495	5	500

IV. *Dorset.*

Sepals.

Petals.	*3*	*4*	*5*	Totals
7	4	—	—	4
8	279	4	—	283
9	115	2	—	117
10	59	4	—	63
11	31	1	—	32
12	3	1	—	4
13	—	1	1	2
Totals	491	13	1	505

B. SEPALS AND STAMENS.

V. *Bordighera.*

Stamens.

Sepals.	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	Totals
2	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
3	1	1	6	4	9	17	20	46	38	85	65	61	77	45	27	25	23	8	7	5	4	4	1	1	—	—	—	1	—	—	—	1	—	582
4	—	—	2	—	—	1	1	3	4	5	—	6	9	5	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
Totals	1	1	8	4	9	18	21	49	42	90	65	67	86	51	28	28	23	8	7	5	4	4	1	1	—	—	—	1	—	—	—	1	1	624

VI. *Guernsey.*

Stamens.

Sepals.	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	Totals
3	1	—	—	2	3	1	11	15	29	39	29	38	53	44	43	25	30	31	24	23	22	18	10	6	6	8	3	—	—	1	515
4	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	1	—	—	1	—	—	—	—	—	—	—	1	—	—	5
Totals	1	—	—	2	3	1	11	15	29	40	29	38	53	45	43	25	31	31	24	24	22	18	10	6	6	8	3	1	—	1	520

VII. *Surrey.*

Stamens.

Sepals.	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	Totals
3	—	—	—	1	—	—	6	11	2	14	29	20	34	29	27	42	31	30	41	33	24	23	22	20	13	8	10	1	5	3	2	5	4	2	2	—	1	495
4	—	—	1	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	1	—	—	—	—	—	—	—	5
Totals	—	—	1	1	—	—	6	11	2	14	29	20	35	29	27	42	31	30	41	33	24	23	23	20	13	8	11	1	5	4	2	5	4	2	2	—	1	500

VIII. *Dorset.*

Stamens.

Sepals.	*18*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*	*60*	*61*	Totals
3	1	–	–	–	–	–	–	2	1	1	5	5	7	8	8	14	17	23	20	35	47	36	34	25	29	37	22	29	17	16	18	6	6	3	1	9	3	3	2	1	–	–	–	–	–	–	–	–	–	491
4	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	2	1	–	3	–	–	1	–	–	2	–	1	1	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	13
5	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1
Totals	1	–	–	–	–	–	–	2	1	1	5	5	7	8	8	14	17	23	20	35	48	36	36	26	29	40	22	29	18	16	18	8	6	4	2	9	3	3	2	1	–	–	–	–	–	–	1	–	1	505

C. SEPALS AND PISTILS.

IX. *Bordighera.*

Pistils.

Sepals.	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	Totals
2	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
3	1	—	—	—	1	3	9	24	63	71	75	82	71	50	31	31	24	14	6	7	1	5	2	1	1	—	2	1	—	1	2	1	—	1	—	1	582
4	—	—	—	—	—	—	—	—	5	4	7	3	7	5	2	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	39
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	1	—	—	—	1	3	9	24	68	75	82	85	79	56	33	34	25	15	6	7	1	5	2	1	1	—	2	1	—	1	2	3	—	1	—	1	624

X. *Guernsey.*

Pistils.

Sepals.	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	Totals
3	5	4	10	17	15	28	32	28	28	33	35	30	30	40	23	28	20	16	13	11	19	15	5	5	5	6	4	2	—	2	—	3	—	2	—	—	—	—	1	515
4	—	—	—	—	1	—	—	—	—	1	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	5
Totals	5	4	10	17	16	28	32	28	28	34	35	30	31	40	24	28	20	16	13	11	19	15	5	5	5	6	4	2	—	2	—	4	—	2	—	—	—	—	1	520

XI. *Surrey.*

Pistils.

Sepals.	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	Totals
3	2	4	3	5	10	15	21	26	25	23	34	34	41	31	31	28	31	21	17	12	15	8	13	5	11	4	6	3	3	4	4	—	1	—	2	1	1	495
4	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	5
Totals	2	4	3	5	10	15	22	26	25	23	34	34	41	31	31	28	31	21	17	13	15	9	13	5	11	4	6	4	3	5	4	—	1	—	2	1	1	500

XII. *Dorset.*

Pistils.

Sepals.	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	*53*	*54–56*	*57*	*57–75*	*76*	Totals
3	1	—	1	1	2	1	2	5	8	7	17	15	18	28	28	29	28	26	36	28	19	33	26	15	17	13	14	13	8	9	8	4	7	8	4	2	2	4	1	1	—	—	1	—	1	—	—	—	—	491
4	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	—	—	—	1	—	1	—	—	—	1	—	—	1	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	1	—	—	13
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
Totals	1	—	1	1	2	1	2	5	8	7	17	15	18	29	29	30	29	27	36	28	19	34	26	16	17	13	14	14	8	9	9	4	8	8	5	2	3	4	1	1	—	—	1	—	1	—	1	—	1	505

D. PETALS AND STAMENS.

XIII. *Bordighera.*

Stamens.

Petals.	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	Totals
6	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7	—	—	—	—	—	1	1	—	—	3	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
8	1	1	7	3	7	16	18	45	37	79	63	60	74	40	24	25	22	8	7	3	—	2	—	—	—	—	—	—	—	—	—	—	—	—	542
9	—	—	—	1	2	1	2	3	3	8	1	4	10	11	4	2	1	—	—	1	3	1	1	1	—	—	—	—	—	—	—	—	—	—	60
10	—	—	1	—	—	—	—	1	2	—	1	—	—	—	—	1	—	—	—	—	1	1	—	—	—	—	—	1	—	—	—	1	1	11
11	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	1	8	4	9	18	21	49	42	90	65	67	86	51	28	28	23	8	7	5	4	4	1	1	—	—	—	1	—	—	—	1	1	624

XIV. *Guernsey.*

Stamens.

Petals.	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	Totals
7	—	—	—	—	—	—	1	—	—	1	—	1	—	1	—	—	—	—	1	1	—	—	—	1	—	—	—	—	—	—	7
8	1	—	—	2	3	1	10	15	28	38	28	36	51	38	39	23	28	23	18	19	17	15	4	3	5	5	1	—	—	1	452
9	—	—	—	—	—	—	—	—	1	—	1	1	2	5	4	2	3	8	4	3	4	2	3	2	1	3	1	—	—	—	50
10	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	1	1	1	2	—	—	—	—	—	—	—	7
11	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	—	—	—	1	1	—	—	4
Totals	1	—	—	2	3	1	11	15	29	40	29	38	53	45	43	25	31	31	24	24	22	18	10	6	6	8	3	1	—	1	520

XV. *Surrey.*

Stamens.

Petals.	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	Totals
6	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
7	—	1	—	—	—	1	—	1	1	—	3	2	1	4	2	—	—	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	20
8	—	—	—	—	6	10	2	13	28	19	29	23	25	35	26	28	38	29	22	17	19	16	11	5	8	1	4	2	2	4	3	2	2	—	—	429
9	1	—	—	—	—	—	—	—	—	1	2	3	1	3	2	1	2	4	—	4	3	4	1	3	3	—	1	2	—	—	1	—	—	—	—	—	42
10	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	1	5
11	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	2
Totals	1	1	—	—	6	11	2	14	29	20	35	29	27	42	31	30	41	33	24	23	23	20	13	8	11	1	5	4	2	5	4	2	2	—	1	500

XVI. *Dorset.*

Stamens.

Petals.	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*	*60*	*61*	Totals
7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
8	—	—	—	—	—	—	—	2	1	1	5	5	5	6	6	12	11	17	13	23	35	22	20	14	17	19	10	13	8	4	8	1	2	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	283
9	—	—	—	—	—	—	—	—	—	—	—	—	2	1	1	1	4	5	5	7	5	5	7	6	8	14	9	10	3	7	4	2	3	1	1	2	3	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	117
10	1	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	—	2	5	7	6	4	3	3	3	4	4	3	4	2	1	2	—	1	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	63
11	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	2	3	2	1	3	—	2	2	2	2	2	—	1	1	3	—	2	—	—	—	—	—	—	—	—	—	—	—	1	—	—	32
12	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
13	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2
Totals	1	—	—	—	—	—	—	2	1	1	5	5	7	8	8	14	17	23	20	35	48	36	36	26	29	40	22	29	18	16	18	8	6	4	2	9	3	3	2	1	—	—	—	—	—	—	1	—	1	505

E. PETALS AND PISTILS.

XVII. *Bordighera.*

Pistils.

Petals.	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	Totals
6	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7	—	—	—	—	—	—	—	—	1	1	2	—	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
8	1	—	—	—	1	2	9	23	63	67	73	77	66	45	29	25	25	13	6	6	1	3	1	—	—	—	1	—	—	1	2	1	—	1	—	—	542
9	—	—	—	—	—	1	—	1	2	7	6	7	9	8	4	7	—	1	—	1	—	—	1	1	1	—	—	1	—	—	—	1	—	—	—	1	60
10	—	—	—	—	—	—	—	—	2	—	1	1	—	—	—	2	—	1	—	—	—	2	—	—	—	—	1	—	—	—	—	1	—	—	—	—	11
11	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	—	—	—	1	3	9	24	68	75	82	85	79	56	33	34	25	15	6	7	1	5	2	1	1	—	2	1	—	1	2	3	—	1	—	1	624

XVIII. *Guernsey.*

Pistils.

Petals.	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	Totals
7	—	—	—	—	—	—	—	1	1	—	—	1	1	—	1	—	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	7
8	5	4	10	16	16	27	30	25	24	32	30	29	28	35	18	26	16	13	9	11	14	11	4	3	3	4	3	2	—	1	—	—	—	2	—	—	—	—	1	452
9	—	—	—	—	—	1	1	2	2	2	4	—	2	5	5	2	2	2	4	—	5	4	1	1	2	1	—	—	—	—	2	—	—	—	—	—	—	—	—	50
10	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	1	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	7
11	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	4
Totals	5	4	10	17	16	28	32	28	28	34	35	30	31	40	24	28	20	16	13	11	19	15	5	5	5	6	4	2	—	2	—	4	—	2	—	—	—	—	1	520

XIX. *Surrey.*

Pistils.

Petals.	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	Totals
6	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
7	—	1	—	—	—	—	3	3	2	1	1	—	1	1	1	3	—	—	—	—	1	—	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	20
8	2	3	3	5	9	15	17	23	22	20	29	31	37	27	27	23	28	18	12	12	12	7	12	4	10	4	2	3	2	3	3	—	1	—	2	—	1	429
9	—	—	—	—	1	—	2	—	1	2	3	3	2	3	3	2	3	2	4	—	1	2	—	—	1	—	3	1	1	1	1	—	—	—	—	—	—	42
10	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1	—	5
11	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	2
Totals	2	4	3	5	10	15	22	26	25	23	34	34	41	31	31	28	31	21	17	13	15	9	13	5	11	4	6	4	3	5	4	—	1	—	2	1	1	500

XX. *Dorset.*

Pistils.

Petals.	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54–56	57	58–75	76	Totals
7	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	–	–	–	1	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4
8	1	–	1	–	2	1	2	4	7	6	15	10	15	20	18	15	17	19	20	15	12	14	13	8	10	7	5	3	3	3	4	2	2	3	2	1	1	1	1	–	–	–	–	–	–	–	–	–	–	283
9	–	–	–	1	–	–	–	–	–	–	1	4	2	7	7	8	7	4	8	10	4	9	9	4	4	1	3	7	3	2	4	1	2	1	1	1	–	–	–	1	–	–	1	–	–	–	–	–	–	–	117
10	–	–	–	–	–	–	–	1	1	1	1	–	1	2	1	4	5	3	5	2	2	5	3	4	1	5	2	4	–	–	–	1	2	4	–	–	2	1	–	–	–	–	–	–	–	–	–	–	–	63
11	–	–	–	–	–	–	–	–	–	–	–	1	–	–	1	2	–	–	3	1	–	5	–	–	2	–	4	–	2	4	1	–	1	–	1	–	–	2	–	–	–	–	–	–	1	–	1	–	–	32
12	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	1	–	–	–	1	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4
13	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	1	2
Totals	1	–	1	1	2	1	2	5	8	7	17	15	18	29	29	30	29	27	36	28	19	34	26	16	17	13	14	14	8	9	9	4	8	8	5	2	3	4	1	1	–	–	1	–	1	–	1	–	1	505

F. STAMENS AND PISTILS.

XXI. *Bordighera.*

Pistils.

Stamens.

	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	Totals
16	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
17	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
18	—	—	—	—	1	—	—	1	1	1	1	1	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
19	—	—	—	—	—	1	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	4
20	—	—	—	—	—	—	1	—	1	1	2	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	9
21	—	—	—	—	—	—	1	2	3	4	3	1	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	18
22	—	—	—	—	—	1	—	2	6	1	7	1	2	—	—	—	—	1	—	—	—	1	1	—	—	—	—	—	—	1	—	—	—	—	—	—	21
23	—	—	—	—	—	1	4	2	7	9	8	3	5	5	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	49
24	—	—	—	—	—	—	—	4	7	10	4	7	5	3	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	42
25	—	—	—	—	—	1	—	7	11	14	16	13	9	5	6	4	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	90
26	—	—	—	—	—	—	1	3	11	13	6	11	5	1	5	3	3	1	—	—	1	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	65
27	—	—	—	—	—	—	—	1	9	9	10	9	11	5	3	4	2	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	67
28	—	—	—	—	—	—	1	2	9	6	11	12	24	9	4	3	1	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	86
29	1	—	—	—	—	—	—	—	1	3	6	14	4	11	2	5	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
30	—	—	—	—	—	—	—	—	—	2	4	6	6	5	—	2	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	28
31	—	—	—	—	—	—	—	—	—	1	2	4	2	4	5	3	6	1	—	1	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	28
32	—	—	—	—	—	—	—	—	—	—	1	2	3	5	1	3	4	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
33	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	2	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
34	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
35	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
36	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
37	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	4
38	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
39	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
40	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
41	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
42	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
43	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
44	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
46	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
48	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	1	—	—	—	1	3	9	24	68	75	82	85	79	56	33	34	25	15	6	7	1	5	2	1	1	—	2	1	—	1	2	3	—	1	—	1	624

XXII. *Guernsey.*

Pistils.

Stamens.

	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	Totals
14	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
15	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
16	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
17	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18	—	—	—	—	—	—	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
19	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
20	2	—	—	—	—	1	—	4	—	2	—	—	—	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
21	—	—	1	1	—	3	2	2	3	1	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
22	—	1	2	3	2	8	1	4	2	2	1	2	1	2	1	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
23	1	1	—	4	3	3	6	2	2	4	4	1	3	1	—	2	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	29
24	—	—	1	1	1	4	1	3	3	1	4	5	—	4	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	40
25	—	1	2	2	4	1	3	2	2	3	2	2	6	3	—	—	1	—	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	29
26	—	—	2	1	3	1	3	—	4	5	5	5	3	5	5	3	—	1	—	2	2	—	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	38
27	—	—	1	2	—	1	5	2	6	4	2	1	3	4	1	2	4	3	1	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	53
28	—	—	—	—	1	4	4	—	2	4	3	5	1	2	3	1	4	—	2	1	2	3	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	45
29	—	—	1	1	—	—	3	2	1	—	1	—	5	4	2	1	—	1	—	2	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	43
30	—	1	—	—	1	—	3	2	—	2	1	2	1	4	3	4	1	1	1	—	2	—	1	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	25
31	—	—	—	—	—	3	—	2	1	1	3	1	3	1	—	6	1	3	1	—	—	—	—	—	1	2	1	—	—	—	—	1	—	—	—	—	—	—	—	31
32	—	—	—	—	—	2	1	2	2	2	3	1	—	—	—	2	1	2	2	1	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31
33	1	—	—	—	—	—	—	—	—	2	1	2	2	2	2	1	1	1	1	1	2	1	1	—	—	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	24
34	—	—	—	—	—	—	—	—	—	1	1	2	—	2	1	1	4	—	2	—	—	3	1	1	—	1	—	—	—	—	—	—	—	2	—	—	—	—	—	24
35	—	—	—	1	—	1	—	—	—	—	—	1	1	1	1	1	—	—	1	—	—	1	1	1	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	22
36	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	2	1	—	—	—	1	1	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	18
37	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	2	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	10
38	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	6
39	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	6
40	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	8
41	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
42	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
43	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	5	4	10	17	16	28	32	28	28	34	35	30	31	40	24	28	20	16	13	11	19	15	5	5	5	6	4	2	—	2	—	4	—	2	—	—	—	—	1	520

XXIII. *Surrey.*

Pistils.

	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	Totals
18							1																															1
19								1																														1
20																																						—
21									1																													6
22	1				1		1		1	1																												11
23	1	1	1			1	2	1	1	1				1																							2
24																1																						14
25					1	2	1	3	1	1	2	2	1		1		3																					29
26		1	1		3	3		3	3	4	2	3	2		2	2		3	1																		20
27		1	1		1	2	1	1	1	1	2	1	2	1	2		1		1	1																	35
28				1	3	1	5	3	3	1	1	3	2	2	1	2		1			1																99
29					2	2	2	2	1	3		3	3	1	2	1	3			1																	27
30					1	1	2	1	2		6	3	2	2	4	4	2	1	3		2																42
31					2	2	3	3	4		3	6	5	4	4	2	5	3				1															31
32				1			2	2	1		3	3	1	3	3	2	3	4	1																		30
33					1	1		1	6		3	4	1	4	3	2	4	1	2	1																	41
34									2	3	5	3	5	3	2	5	1	2	1	3	1																33
35								1	1	1	3	3	5	2	3	1	2	1	1	1	2																	24
36								1	1	1	1	1	4	1	4	1	2	1	1	1	1	2																23
37												1	3		1	3	2	2	3	1	1	1	1														23
38												1			1	2	2	3	2	1	1	1	1	1													20
39																1	2	1	1	1	3	1	1	1	1												13
40																						1		1														8
41																							2															11
42																							1															1
43																																						5
44																																						4
45																																						2
46																																						5
47																																						4
48																																						2
49																																						2
50																																						—
51																																						1
52																																						
Totals	2	4	3	5	10	15	22	26	25	23	34	41	31	31	28	31	36	37	38	39	40	41	42	43	44	45	46	Totals										500

Stamens.

XXIV. *Dorset.*

Pistils.

Stamens.

	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	No reading 58—75	76	Totals
13	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
14	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
15	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
16	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
17	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
18	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
19	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2
20	1	–	1	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
21	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
22	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	5
23	–	–	–	–	–	–	1	–	–	–	2	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	5
24	–	–	–	–	–	–	–	1	–	2	–	–	–	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	7
25	–	–	–	1	–	–	–	1	–	–	3	–	–	1	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	8
26	–	–	–	–	–	–	–	1	–	1	1	1	–	2	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	8
27	–	–	–	–	–	–	–	1	–	–	1	–	–	1	–	1	1	–	2	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	14
28	–	–	–	–	–	–	–	–	–	1	2	1	1	3	1	–	–	–	2	1	–	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	17
29	–	–	–	1	–	–	–	1	1	–	–	1	2	1	3	2	1	–	½	1	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	23
30	–	–	–	–	–	–	–	1	2	1	1	1	–	2	3	3	1	3	–	–	–	1	2	–	–	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	20
31	–	–	–	–	–	–	–	–	1	–	–	4	1	2	5	1	2	–	–	–	–	3	–	–	–	–	–	1	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	35
32	–	–	–	–	–	–	–	–	2	1	1	–	4	1	6	1	2	4	1	3	3	2	2	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	48
33	–	–	–	–	–	–	–	–	1	1	–	3	4	4	5	2	2	6	3	5	1	3	5	2	2	–	2	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	37
34	–	–	–	–	1	–	–	–	–	–	1	1	2	2	1	5	2	1	4	3	2	3	2	2	–	2	1	–	2	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	35
35	–	–	–	–	–	–	1	–	–	–	1	1	1	2	1	2	1	2	2	2	2	2	1	5	1	–	1	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	26
36	–	–	–	–	–	–	–	–	–	2	1	–	–	2	1	2	2	1	4	1	4	1	1	1	2	1	1	–	1	–	–	1	–	–	–	–	–	1	–	1	–	–	–	–	–	–	–	–	–	–	29
37	–	–	–	–	–	–	–	–	–	–	–	–	1	3	1	2	2	1	6	1	3	1	2	1	4	2	3	1	4	–	2	–	–	2	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	40
38	–	–	–	–	–	–	–	–	–	–	1	2	–	–	1	2	1	6	1	–	4	1	2	4	1	2	1	1	–	1	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	22
39	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4	1	1	2	1	4	3	–	2	–	3	–	1	–	–	2	1	3	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	29
40	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	1	2	2	1	3	–	–	1	–	1	2	–	2	1	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	18
41	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	1	–	–	–	1	2	–	3	2	1	–	2	–	–	1	1	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	16
42	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	–	–	1	–	–	1	1	–	–	1	1	–	3	1	–	1	–	–	–	–	–	–	–	–	–	–	–	–	18
43	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	3	–	–	2	–	–	–	1	–	–	1	1	–	1	1	–	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	8
44	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	1	1	–	–	–	–	–	–	1	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	6
45	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4
46	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2
47	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	–	1	–	1	–	–	–	1	1	–	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	9
48	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	3
49	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	3
50	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2
51	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	1
52	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
53	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
54	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
55	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
56	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
57	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
58	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	1
59	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
60	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1
61	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
Totals	1	–	1	1	2	1	2	5	8	7	17	15	18	29	29	30	29	27	36	28	19	34	26	16	17	13	14	14	8	9	9	4	8	8	5	2	3	4	1	1	–	–	1	–	1	–	–	–	1	–	1	505

SECOND REPORT ON THE RESULT OF CROSSING JAPANESE WALTZING MICE WITH EUROPEAN ALBINO RACES.

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THE present paper is an account of the continuation of work, the first part of which was published in the last number of this Journal. It contains, first, an additional record of the results of crossing Japanese waltzing mice with albino mice, embodying the previously recorded nine families and adding eleven new ones; and secondly the result of pairing hybrids resulting from such crosses, and of crossing these hybrids with albinos.

Crosses between Japanese Waltzing and Albino Mice.

The number of young obtained from such crosses has increased since the last paper from 48 to 154; but the uniformity of the result (the almost universal presence of patches of colour like that of the house-mouse) has not been maintained: there have appeared besides more yellow mice, black mice, and black and white.

The classification of colours in the hybrid has therefore had to be re-organized. The hybrids nearly always shew a considerable amount of white: they are now classified according to the amount of whiteness into five groups: each group is then subdivided into five classes according to the colour itself.

Group 1 (Fig. 1) has more white and less extent of coloured patches than the normal waltzing mouse. The distribution of colour on a waltzer is shewn in Fig. 6.

Group 2 (Fig. 2) has about as much white as a normal waltzing mouse.

Group 3 (Fig. 3) has less white (i.e. greater extent of coloured patches) than a waltzer.

Group 4 (Fig. 4) has still less white and leads on to

Group 5 (Fig. 5) which has no pure white: but the belly is whitish not gray.

Group 6 contains mice whose bellies are nearly the same colour as their backs. A gray mouse of group 6 (6*c* or 6*d*) is therefore indistinguishable from a house-mouse.

FIG. 1.

FIG. 2.

FIG. 3.

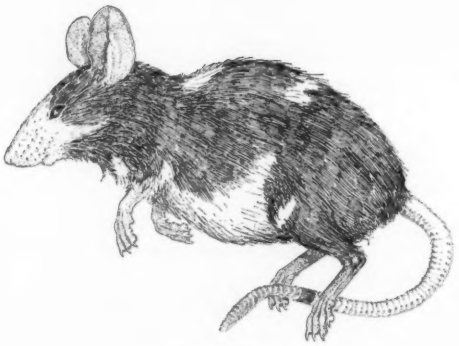


FIG. 4.

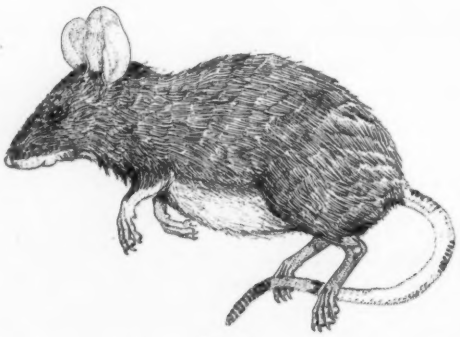


FIG. 5.

FIG. 6.

The individuals of each group are then classified according to the colour of the coloured patches.

Class a = yellow.
Class b = fawn yellow.
Class c = pale wild colour } (i.e. that of the house-mouse).
Class d = dark wild colour }
Class e = black,
and *Class* f = "lilac" = pale blue gray; at present only exhibited by the offspring of hybrids.

It is not suggested that these form one colour series. Table I. shews the distribution of colour and the colour itself in all the individuals of twenty families produced by crossing waltzing with albino mice. All the mice in the Table have black eyes.

TABLE I.

Number of Cross	Group I.					Group II.					Group III.					Group IV.					Group V.					Group VI.					Totals
	a	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	
i	–	–	2	–	–	–	–	1	1	–	–	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	6
ii	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
vii	–	–	–	–	–	–	–	2	1	–	–	–	1	1	–	–	–	–	–	–	–	–	2	–	–	–	–	–	–	–	7
viii	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	6	–	–	–	–	–	–	–	6
ix	–	–	–	–	–	–	–	5	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	6
xii	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	3	–	–	–	–	–	–	5
xiii	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	–	–	–	–	–	–	1	1	–	–	–	–	–	–	4
xvi	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	3
xx	–	–	–	–	–	–	–	1	2	–	–	–	–	1	–	–	–	–	–	–	–	–	–	2	–	–	–	–	–	–	6
xxvii*	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
xxxiii*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	1
xxxv	–	–	–	–	–	–	–	1	1	–	–	–	–	2	–	–	–	–	2	–	–	–	–	–	–	–	–	–	–	–	6
xlv	–	–	–	–	–	–	–	4	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	.	–	4
lix*	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	2
lxi	–	–	–	–	–	–	–	1	–	–	–	–	1	–	–	–	–	1	–	–	–	–	4	–	–	–	–	–	–	–	7
lxii	–	–	–	–	–	–	–	1	–	–	–	–	2	–	–	–	–	–	–	–	–	–	3	–	–	–	–	–	–	–	6
lxxiii	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	–	–	–	–	–	–	4	–	–	–	–	–	–	–	6
lxxiv	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	6	–	–	–	–	–	–	–	–	–	–	–	–	7
lxxv	–	–	1	–	–	–	–	2	–	–	–	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	5
lxxvii	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2	–	1	–	–	4	–	1	–	–	–	–	–	8
lxxviii	–	–	–	–	–	–	–	–	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	1	1	–	–	–	–	–	4
lxxxii*	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	2
lxxxiv	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	5	–	4	–	–	–	–	–	–	–	9
lxxxv	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	1	–	–	–	–	3	1	–	–	–	–	–	7
lxxxvi	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	–	–	–	–	–	–	5	–	–	–	–	–	–	–	7
xciv*	–	–	–	–	–	–	–	–	–	–	–	–	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2
c	–	–	–	–	–	–	–	1	–	–	–	–	3	–	–	–	–	–	–	–	–	–	5	–	–	–	–	–	–	–	9
ci	–	–	–	–	–	–	–	1	–	–	–	–	2	–	–	–	–	1	–	–	–	–	5	–	–	–	–	–	–	–	9
ciii	–	–	–	–	–	–	–	1	–	–	–	–	3	–	–	–	–	–	–	–	–	–	4	–	–	–	–	–	–	–	8
Totals	–	–	3	–	–	–	–	21	7	2	–	–	27	5	3	–	–	12	3	1	7	–	49	10	3	–	–	–	1	–	154

* In these crosses the parents were, ♀ waltzing mouse × ♂ albino; in all the rest the parents were ♀ albino × ♂ waltzer.

From the following re-classification of the above Table I. the frequency of degrees of whiteness among 154 hybrids will be at once evident.

Group 1	3
Group 2	30
Group 3	35
Group 4	16
Group 5	69
Group 6	1
	154

That is to say the majority of hybrids have coloured patches in, roughly, the same place as the waltzers have; *or* are wild-coloured mice except for the belly.

The following arrangement shews the frequency of colours among 154 hybrids.

Class a.....	7
Class b.....	0
Class c	112
Class d.....	26
Class e	9
	154

That is to say by far the greater number of hybrids exhibit wild colour.

Offspring of Hybrids.

Hybrids have been paired *inter se* and crossed with albinos.

A. *Offspring of pairs of Hybrids.*

The number of mice which waltz in this generation is eight. At present nine albinos have appeared: the actual arrangement of the colours and the colours themselves are here shewn: the letter *w* indicates that the mouse exhibits waltzing movements, and *p* that it has pink eyes. It is unnecessary to say that albinos have pink eyes.

H_7. ♀ 2*d* × ♂ 2*c*.
3 *Young.* 2*f* (*wp*), 2*a* (*p*), 2*b*.

H_8. ♀ 1*c* × ♂ 2*c*.
4 *Young.* 3*b* (*p*), 3*c*, 4*a* (*p*), 4*a* (*p*).

H_9. ♀ 2*d* × ♂ 3*c*.
5 *Young.* 3*a*, 3*b*, 3*b*, 3*e* (*w*), 3*f*.

H_{18}. ♀ 2*c* × ♂ 2*c*.
7 *Young.* 4 *albinos*, 3*b* (*wp*), 3*b* (*p*), 3*c*.

H_{19}. ♀ 2*c* × ♂ 3*c*.
7 *Young.* 2 *albinos*, 2*c*, 2*c*, 2*c*, 2*c* (*w*), 4*e* (*w*).

H_{20}. ♀ 2*c* × ♂ 2*c*.
6 *Young.* *Albino.* *Albino* (*w*), 2*c* (*w*), 3*c* (*w*), 2*e*, 2*f*.

H_{29}. ♀ 2*c* × ♂ 3*c*.
5 *Young.* *Albino*, 2*e*, 2*c*, 4*e*, 4*d*.

That is to say there are 37 young from 14 hybrid parents of degrees of whiteness included in Groups 2 and 3. The young may be classified according to degrees of whiteness and according to their colour.

Degrees of Whiteness		Nature of Colour	
Albinos	9	*Albinos*	9
Group 1	0	*Class a*	4
Group 2	12	*Class b*	6
Group 3	11	*Class c*	9
Group 4	5	*Class d*	1
	37	*Class e*	5
		Class f	3
			37

B. *Offspring of Hybrids and Albinos.*

The following is a record of young produced by crossing a hybrid with an albino.

H_1. ♀ albino × ♂ 3*c*.
4 *Young.* 5*c*, 5*c*, 5*c*, 5*c*.

H_2. ♀ albino × ♂ 3*c*.
9 *Young.* 5 *albinos* and 4*e*, 4*d*, 4*d*, 5*d*.

H_4. ♀ albino × ♂ 5*c*.
5 *Young.* 2 *albinos* and 5*c*, 5*d*, 6*e*.

H_5. ♀ albino × ♂ 5*c*.
5 *Young.* 1 *albino* and 6*d*, 6*d*, 6*d*, 6*c*.

H_6. ♀ albino × ♂ 5*c*.
4 *Young.* 2 *albinos* and 6*d*, 6*d*.

H_{11}. ♀ albino × ♂ 3*d*.
6 *Young.* 5 *albinos* and 6*e*.

H_{12}. ♀ 2*c* × ♂ albino.
5 *Young.* 2 *albinos* and 3*c*, 6*e*, 6*e*.

H_{13}. ♀ 5*c* × ♂ albino.
3 *Young.* 1 *albino* and 5*d*, 6*e*.

H_{14}. ♀ 5*c* × ♂ albino.
6 *Young.* 4 *albinos* and 2*c*, 3*e*.

H_{15}. ♀ albino × ♂ 5*c*.
8 *Young.* 4 *albinos*, 3*a*, 3*a*, 6*a*, 3*e*.

H_{17}. ♀ albino × ♂ 5*c*.
5 *Young.* 2 *albinos*, 3*e*, 3*d*, 6*d*.

H_{24}. ♀ albino × ♂ 5*c*.
6 *Young.* 3 *albinos*, 4*d*, 4*d*, 6*d*.

H_{26}. ♀ albino × ♂ 3*c*.
4 *Young.* 1 *albino*, 6*e*, 5*d*, 5*d*.

H_{28}. ♀ 5*c* × ♂ albino.
2 *Young.* 5*d*, 5*c*.

H_{31}. ♀ 5*d* × ♂ albino.
1 *Young.* 1 *albino*.

H_{32}. ♀ 5*c* × ♂ albino.
6 *Young.* 2 *albinos* and 6*c*, 6*c*, 4*c*, 6*e*.

H_{33}. ♀ albino × ♂ 5*c*.
9 *Young.* 4 *albinos* and 6*e*, 6*e*, 6*e*, 6*e*, 4*c*.

That is to say there is a fairly sharp segregation into albino and wild-coloured mice. Out of 88 young there are 39 albino, 31 wild-coloured, 15 black and 3 yellow mice.

Discussion of Results.

1. *The First Generation.*

As Table I. shews, the first generation has not a uniform colour: any modification of Mendel's hypothesis involves the uniformity of the first generation. For it is one of the Mendelian principles that segregation of characters does not occur until the generation produced by pairing individuals of the first (hybrid) generation. The generation, thus produced, essentially consists of 25 °/₀ organisms with the recessive character and 75 °/₀ with the dominant; and this not only in the case of a simple character, but in that of a complex one; for example the hybrids produced by Mendel's crossing a white flowered with a purple flowered (a complex colour according to Mendel) bean were all purple. But the offspring produced by the pairing of these purple hybrids exhibited the most marked heterogeneity. The essential point is that segregation of however complex characters never occurs before the second generation*.

Now in these mice this segregation seems to take place in the first generation, for there appear in it besides the wild colour, yellow and black, as can be seen from Table I.

* G. Mendel: "Versuche mit Pflanzen-Hybriden," *Verhandl. des Naturforsch. Ver. Brünn*, IV. Band. 1865 (Abhandlungen), pp. 34, 35.

The view that albinism is "recessive" in Mendel's sense implies that albinos of any ancestry, provided they themselves are really albino, will behave in the same way when crossed. Now, the young of all the albinos are not the same and at the end of my last paper it was pointed out that a difference in the kind of the litter corresponded to a difference in the ancestry of the albino mother: that is to say, that wild-coloured mice only appeared in the litters of pure-bred albinos: and the generalization that the more in-bred a mouse was, the less power it had of transmitting its whiteness, was suggested. The evidence brought forward for this were the results of von Guaita who in four crosses between waltzing mice and albinos which were in-bred for 29 generations always got absolutely wild-coloured mice; and my own results consisting of nine such crosses. This generalization is borne out by the additional eleven litters which have appeared since my last paper was written; as a classification of litters into (*a*) those from cross-bred albinos and (*b*) those from pure-bred albinos will readily shew (Table II.).

TABLE II.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
(*a*) Young from cross-bred albinos	2	11	8	4	4	—
(*b*) Young from pure-bred albinos	—	13	23	—	60*	1

In my previous paper I think I did not sufficiently emphasize the difference which seems to me to exist between in-bred and pure-bred mice: the above table is certainly of pure-bred mice; but what is not certain is whether the *relative* inability to transmit whiteness among pure-bred mice is due to the fact of their being pure-bred, or in-bred. Pure-bred mice usually are in-bred and von Guaita's unquestionably were; but this does not help us to decide whether the relative inability to transmit whiteness is due to in-breeding or pure-breeding; it only shews that it is not impossible that it may be due to in-breeding. The meanings of these terms should be clearly understood; there are two pairs of categories, one pair in which cross-bred is antithetic to pure-bred, and another pair in which out-bred is antithetic to in-bred, the former pair referring to the presence or absence of any other colour than white in the ancestry of the mouse, the latter pair to the distance or nearness of relation of the parents, grandparents, etc. of the mouse under consideration. And before the cause of the great preponderance of the number of wild-coloured mice in the litters of pure-bred over the number of those in the litters of cross-bred albinos can be ascertained waltzing mice must be crossed with in-bred and out-bred pure-bred albinos and in-bred and out-bred cross-bred albinos. (Such kinds of albinos are being reared as quickly as possible.) Whatever the explanation be, the fact remains that the ancestry of a white mouse does make a difference in the character of its offspring when crossed with a waltzing mouse.

* The 60 in the table includes the 7 yellow mice 5*a* in which there is no whiteness.

2. *The Second Generation.*

It will be seen by inspection of the tables recording the offspring of hybrids, that when a hybrid is paired with a hybrid the offspring form a much more heterogeneous collection than does the offspring from a hybrid and a white in which the young are more sharply segregated into white and wild-coloured. And this difference is coincident with, if not causally connected with a difference in the ancestry of the two sets of young. For the heterogeneous offspring of two hybrids have a more complex ancestry than do the less variable offspring of a hybrid and an albino; and it may be a fact of similar meaning that the hybrids produced by crossing a waltzing mouse with a *cross-bred* albino are more heterogeneous than those produced by a similar cross in which the albino, however, was *pure-bred* (see Table II.)—in both cases the more heterogeneous collection of offspring comes from parents of which the ancestry is more complex; that is to say the young of the cross-bred albinos are more heterogeneous than those of the pure-bred; and the young produced by pairing two hybrids are more heterogeneous than those produced by crossing hybrids with albinos.

No theory of compound allelomorphs such as that put forward by Mendel will account for this striking difference between the variability of the two groups of offspring.

Remembering that 6*c* or 6*d* indicates a mouse indistinguishable from the common *Mus musculus*, the curious fact will be noticed that only *one* such appeared among 154 hybrids of the first generation, whereas out of the 88 offspring of a cross between a hybrid and an albino there were 10. This result may be parallel with that obtained by Darwin* when he crossed a white fantail with a black barb and also a barb with a spot and then crossed the mongrel bard fantail with the mongrel barb spot and got "a bird of as beautiful a blue colour, with the white loins, double black wing-bar, and barred and white-edged tail-feathers, as any wild rock-pigeon!"; i.e. reversion, in both cases, did not occur until the third generation.

The facts so far observed which are in possible accordance with some form of Mendelian hypothesis are (1) the apparently regular appearance of albinos when hybrids are crossed with albinos, although the evidence at present available does not suffice to shew whether these occur in Mendelian proportions (50 % albino and 50 % hybrid) or not, (2) the well-known fact that albinos of any ancestry when paired together produce albino young, exceptions to this rule being at least very rare, and (3) the appearance of waltzing and albino mice in the second hybrid generation.

The first of these results although not inconsistent with the truth of Mendel's hypothesis cannot be taken as proof that this hypothesis applies; for a similar result is observed in such cases as that of human eye-colour where Mendel's Laws have been shewn not to hold†.

* *The Origin of Species*, pp. 17, 18.

† Karl Pearson: see below, pp. 213 *et seq.*

Postscript, added Feb. 12, 1903.

The mice, resulting from a first cross between albinos and pink-eyed waltzing mice, are now 203, all the individuals having dark eyes, while none have wholly white fur, and none waltz. In the second generation, there are 66 mice produced by pairing hybrids; of these 13 are pink-eyed albinos, and 17 have pink eyes and coats more or less coloured. In the same generation there are 205 individuals produced by crossing albinos with hybrids; of these 111 are albinos, the remainder having dark eyes and some colour in the coat.

The proportions of albinos and of individuals not albino are not in disagreement with Mendel's results, but the exhibition of these results in Mendelian form depends on the adoption of a quite artificial category of coat-colours; for albinism includes only pure whiteness of coat, while coloured coats include the whole range of conditions from white with small patches of pale yellow to dark "wild colour" or black.

The inheritance of eye-colour is not in accordance with Mendel's results. For since pink eyes occur in particoloured mice, the possession of pink eyes must on Mendel's view depend on a separate embryonic element from that which determines coat-colour. Pink eyes are however not "dominant," since the two pink-eyed parents of the first generation always produce dark-eyed young. For the same reason pink eyes are not "recessive." Yet although pink eyes disappear in the first generation (the result of crossing two pink-eyed parents) they reappear in the second; but a correlation is then established between coat-colour and eye-colour which is strong in the offspring of hybrids paired together, and at present perfect in the offspring of hybrids and albinos. The behaviour of eye-colour is thus in every respect discordant with Mendel's results.

NEW TABLES OF THE PROBABILITY INTEGRAL*.

By W. F. SHEPPARD, M.A., LL.M.

Description of the Tables.

1. The "probability integral" expresses the area of the normal curve, or curve of error, whose equation is

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The abscissa x is measured from the central ordinate, about which the curve is symmetrical, and its unit of measurement is the standard deviation (square root of mean square of deviation). The whole area of the curve is unity. If this whole area be divided by the ordinate z, at distance x from the central ordinate, into portions $\frac{1}{2}(1+\alpha)$ and $\frac{1}{2}(1-\alpha)$, then

$$\tfrac{1}{2}(1+\alpha) = \int_{-\infty}^{x} z\,dx, \quad \tfrac{1}{2}(1-\alpha) = \int_{x}^{\infty} z\,dx \quad \ldots\ldots\ldots\ldots\ldots (2),$$

$$\alpha = 2\int_{0}^{x} z\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{A}),$$

where z has the value given by (1).

Tables I. and II. give the values of z and of $\frac{1}{2}(1+\alpha)$ for any value of x from ·00 to 6·00; and therefore, the curve being symmetrical about the central ordinate, they enable us to determine the values of $\frac{1}{2}(1+\alpha)$, $\frac{1}{2}(1-\alpha)$, and z, for any value of x between $-6{\cdot}00$ and $+6{\cdot}00$. Tables III. and IV. give the values of x and of z for any value of α from ·00 to ·80; and therefore for any value of α between $-{\cdot}80$ and $+{\cdot}80$.

If X denotes the measurement of an organ common to a large number N of individuals, and if the different values of X are distributed according to the normal law about a mean value m with mean square of deviation σ^2, then the

* [In the prospectus of *Biometrika*, the Editors promised to provide "numerical tables tending to reduce the labour of statistical arithmetic." The first instalment of such tables by Mr Palin Elderton was given in Vol. I. This second instalment provides for the widely felt want of probability integral and normal curve data calculated on the basis of the *standard deviation* and not on that either of the modulus or of the probable error. Editors.]

numbers of individuals whose measurements are respectively less and greater than any particular value X are N_1 and N_2, where

$$N_1 = \tfrac{1}{2}N(1+\alpha), \quad N_2 = \tfrac{1}{2}N(1-\alpha),$$

and the ordinate of the figure of frequency corresponding to the measurement X is

$$Nz/\sigma,$$

where $\frac{1}{2}(1+\alpha)$ and z are the values of the area and of the ordinate given by Tables I. and II. as corresponding to

$$x = (X-m)/\sigma.$$

Method of Using.

2. *Interpolation.* If a quantity u be tabulated in terms of x, and u_0 be the value of u for $x = x_0$, then its value for $x = x_0 + \theta h$, where h is the common difference of x, and θ is less than 1, is usually found from u_0 and its advancing differences by the formula

$$u = u_0 + \theta\Delta u_0 - \frac{\theta(1-\theta)}{2!}\Delta^2 u_0 + \frac{\theta(1-\theta)(2-\theta)}{3!}\Delta^3 u_0 - \dots \dots \quad (3).$$

An alternative formula, in which central differences are used, is that recently given* by Professor Everett. When we do not require to go beyond third differences, this formula is most conveniently written

$$u = u_0 + \theta\Delta u_0 - \frac{\theta(1-\theta^2)}{3!}\delta^2 u_1 - \frac{\phi(1-\phi^2)}{3!}\delta^2 u_0 \quad \dots\dots\dots\dots\dots (4),$$

where $\delta^2 u_1$ and $\delta^2 u_0$ are the second central differences of u_1 and u_0 (i.e. $\delta^2 u_1 = \Delta^2 u_0$, $\delta^2 u_0 = \Delta^2 u_{-1}$), and $\phi = 1-\theta$.

In practice, when second and higher differences have to be considered, and a mechanical method of performing multiplications is available (e.g. by means of a Brunsviga), it is more convenient to use differential coefficients. Let the means of the pairs of odd differences above and below the line of u_0 be denoted by $\mu\delta u_0$, $\mu\delta^3 u_0$, ... (i.e. $\mu\delta u_0 = \frac{1}{2}(\Delta u_0 + \Delta u_{-1})$, $\mu\delta^3 u_0 = \frac{1}{2}(\Delta^3 u_{-1} + \Delta^3 u_{-2})$, ...), and let Du_0, D^2u_0, D^3u_0, ... be the values for $x = x_0$ of the successive differential coefficients of u with regard to x; then†, δ^2 having the same meaning as in (4),

$$\left.\begin{aligned} hDu_0 &= \mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \dots \\ h^2D^2u_0 &= \delta^2 u_0 - \tfrac{1}{12}\delta^4 u_0 + \dots \\ h^3D^3u_0 &= \mu\delta^3 u_0 - \tfrac{1}{4}\mu\delta^5 u_0 + \dots \\ h^4D^4u_0 &= \delta^4 u_0 - \dots \\ &\text{etc.} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots (5),$$

and, for $x = x_0 + \theta h$,

$$u = u_0 + \theta\left[hDu_0 + \tfrac{1}{2}\theta\left\{h^2D^2u_0 + \tfrac{1}{3}\theta\left(h^3D^3u_0 + \dots\right)\right\}\right] \quad \dots\dots\dots\dots\dots (6).$$

* *Journal of the Institute of Actuaries*, Vol. xxxv. p. 452.

† *Proc. of the Lond. Math. Soc.* Vol. xxxi. p. 465.

We need only use inside the square brackets the numerical values of θ and of hDu_0, $h^2D^2u_0$, ..., if we put the sign of $\theta . hDu_0$ outside, and follow the rule of signs inside. The rule is that a *minus* sign must be inserted, if θ be positive, whenever there is a change of sign in the series hDu_0, $h^2D^2u_0$, ..., or, if θ be negative, whenever there is no change of sign in this series.

Thus, in Table I., taking u to be $\frac{1}{2}(1+\alpha)$, we have for $x_0 = \cdot 40$ (omitting decimal point)

$\mu\delta u_0$	$\delta^2 u_0$	$\mu\delta^3 u_0$
$36826\frac{1}{2}$	-147	-3
hDu_0	$h^2D^2u_0$	$h^3D^3u_0$
36827	-147	-3

and therefore, for $x = \cdot 40 + \cdot 01\theta$,

$$\tfrac{1}{2}(1+\alpha) = 10^{-7}(6554217 + \theta[36827 - \tfrac{1}{2}\theta\{147 + \tfrac{1}{3}\theta . 3\}]).$$

For $x = \cdot 40 - \cdot 01\theta$ we should replace $+\theta$, $-\frac{1}{2}\theta$, $+\frac{1}{3}\theta$, by $-\theta$, $+\frac{1}{2}\theta$, $-\frac{1}{3}\theta$. The formulæ given by (3) and (4) would be, for $x = \cdot 40 + \cdot 01\theta$,

$$\tfrac{1}{2}(1+\alpha) = 10^{-7}\left(6554217 + 36753\theta + 150\,\frac{\theta(1-\theta)}{2} - 3\,\frac{\theta(1-\theta)(2-\theta)}{6}\right),$$

$$\tfrac{1}{2}(1+\alpha) = 10^{-7}\left(6554217 + 36753\theta + 150\,\frac{\theta(1-\theta^2)}{6} + 147\,\frac{\phi(1-\phi^2)}{6}\right).$$

There is no difficulty about the divisions represented by the coefficients $\frac{1}{2}$, $\frac{1}{3}$, ... of θ in (6); but, if we wish to avoid them, we may calculate $h^2D^2u_0/2!$, $h^3D^3u_0/3!$, ... and write the formula

$$u = h_0 + \theta[hDu_0 + \theta\{\tfrac{1}{2}h^2D^2u_0 + \theta(\tfrac{1}{6}h^3D^3u_0 + \ldots)\}] \quad \ldots\ldots\ldots\ldots\ldots(7).$$

3. *Inverse Interpolation.* In most cases N_1 and N_2 [$= \frac{1}{2}N(1+\alpha)$ and $\frac{1}{2}N(1-\alpha)$] are known, and we require x. If $\frac{1}{2}(1+\alpha)$ and $\frac{1}{2}(1-\alpha)$ are both less than $\cdot 90$, we can use Table III., which gives x in terms of $\alpha = (N_1 - N_2)/N$. But, if either $\frac{1}{2}(1+\alpha)$ or $\frac{1}{2}(1-\alpha)$ be greater than $\cdot 90$, we must use Table I., by inverse interpolation. By (6) we have, if $x = x_0 + \theta h$,

$$\theta = (u - u_0) \div [hDu_0 + \tfrac{1}{2}\theta\{h^2D^2u_0 + \tfrac{1}{3}\theta(h^3D^3u_0 + \ldots)\}] \quad \ldots\ldots\ldots\ldots\ldots(8),$$

or, if $x = x_0 - \theta h$,

$$\theta = (u_0 - u) \div [hDu_0 - \tfrac{1}{2}\theta\{h^2D^2u_0 - \tfrac{1}{3}\theta(h^3D^3u_0 - \ldots)\}] \quad \ldots\ldots\ldots\ldots(8A).$$

The value of θ, and thence that of x, is obtained by successive approximations.

Suppose, for instance, that

$$\tfrac{1}{2}(1+\alpha) = \cdot 654.$$

If $x = \cdot 40 - \cdot 01\theta$, we have from Table I.

$$\theta = 14217 \div [36827 + \tfrac{1}{2}\theta\{147 - \theta\}].$$

A first approximation gives

$$\theta = 14217 \div 36827 = \cdot 38605;$$

and, with this value, the corrected divisor becomes

$$36855 \cdot 2,$$

which gives for a second approximation

$$\theta = \cdot 38575,$$

and therefore

$$x = \cdot 3961425.$$

The correct value, as given by Table III., is

$$x = \cdot 3961424.$$

The degree of accuracy with which x can be obtained by this method depends on the relation of magnitude between the differences of x and of u. In the above example, to a difference of ·01 in x there corresponds a difference of very little more than one-third of ·01 in $\frac{1}{2}(1+\alpha)$; and therefore, if x is calculated from Table I. to seven places of decimals, it will only be accurate within about 2 in the last figure. The possible inaccuracy of x increases as $\frac{1}{2}(1+\alpha)$ increases. But this is not important, as the "probable error" of x, for any given number of observations, also increases.

4. *Smoothing.* In arranging a table, with differences, for the calculation of any quantity u, it is usual to enter in the difference-columns the actual (or "tabular") differences of the values of u as tabulated. In the present tables I have adopted a different method, and have given the differences as near as possible to the differences of the true values of u. The object of this is to enable greater accuracy to be obtained when required. If we only want u to five or six places of decimals, it is immaterial whether we use the tabular or the corrected differences. But, if we wish to have it as accurate as possible, we can alter the tabulated values by inspection.

Looking, for instance, at the commencement of Table I., it is clear that the tabulated values of $\frac{1}{2}(1+a)$ are too great for $x = \cdot 01$ and $x = \cdot 03$, while they are too small for $x = \cdot 02$ and $x = \cdot 04$. Taking $u = \frac{1}{2}(1+a) \times 10^7$, so as to omit the decimal point, the table may be written

x	u	Δ +	Δ^2 −
·00	5000000		0
		$39894 - \theta$	
·01	$5039894 - \theta$		$4 + (1 - 2\theta - \phi)$
		$39890 - (1 - \theta - \phi)$	
·02	$5079783 + \phi$		$8 - (1 - \theta - 2\phi - \chi)$
		$39882 - (\phi + \chi)$	
·03	$5119665 - \chi$		$12 + (1 - \phi - 2\chi - \psi)$
		$39870 - (1 - \chi - \psi)$	
·04	$5159534 + \psi$		
⋮	⋮	⋮	⋮

By tabulating u by differences of ·05 or ·10 in x, it will be found* that the third difference in Table I., for these values of x, is almost exactly 4. We see therefore that $\theta + \phi$ and $\chi + \psi$ are both greater than $\frac{1}{2}$, and $\phi + \chi$ is less than $\frac{1}{2}$; while θ, $1 - \theta - \phi$, $\phi + \chi$, and $1 - \chi - \psi$ are all very nearly equal. The values $\theta = \cdot 4$, $\phi = \cdot 2$, $\chi = \cdot 2$, $\psi = \cdot 4$ satisfy these conditions; and, as a matter of fact, they give for $\frac{1}{2}(1+a)$ values which are correct within 1×10^{-8}.

* For the relations between differences of u for large and small differences in x, see *Proc. Lond. Math. Soc.* Vol. XXXI. pp. 468—471.

Examples of Application.

5. *Expression of data in terms of x.* When a distribution is nearly normal, we may state the data by expressing X (see § 1) in terms of x. For an example, take the head-breadths of 3000 criminals, given on p. 214 of *Biometrika*, Vol. I. The interval in X is ·1 of a centimetre: but, for brevity, we shall take intervals of ·3 of a centimetre.

We should first note the probable errors. If, of the N values, N_1 lie below X and N_2 above it, the probable error in N_1 or N_2 is $\pm Q\sqrt{N_1N_2/N}$, where $Q = ·67449$; i.e. it is an even chance that N times the true proportion of values below X lies between $N_1 - Q\sqrt{N_1N_2/N}$ and $N_1 + Q\sqrt{N_1N_2/N}$. Thus, for $X = 14·55$ cm., the P. E. is $\pm Q\sqrt{479 \times 2521 \div 3000} = 13·5$. Calculating the probable errors, the data may be expressed thus:—

X	N_1	N_2	P. E.	X	N_1	N_2	P. E.
13·35	0	3000	—	*15·45*	2364	636	±15·1
13·65	8	2992	± 1·9	*15·75*	2763	237	±10·0
13·95	38	2962	± 4·1	*16·05*	2925	75	± 5·8
14·25	157	2843	± 8·2	*16·35*	2978	22	± 3·2
14·55	479	2521	±13·5	*16·65*	2997	3	± 1·2
14·85	1077	1923	±17·7	*16·95*	3000	0	—
15·15	1762	1238	±18·2				

Now calculate the values of x and of z corresponding to $N_1 = \frac{1}{2}N(1+\alpha)$. An error of θ in x is equivalent to an error of $z\theta$ in $\frac{1}{2}(1+\alpha)$, so that the above values of the P. E. have to be divided by $3000z$ to give the P. E. in x. We thus get the data in the form:—

X	x	P. E.	X	x	P. E.
13·35	$-\infty$	—	*15·45*	+ ·800	±·017
13·65	−2·786	±·077	*15·75*	+1·412	±·023
13·95	−2·236	±·042	*16·05*	+1·960	±·033
14·25	−1·625	±·020	*16·35*	+2·441	±·052
14·55	− ·996	±·019	*16·65*	+3·090	±·135
14·85	− ·361	±·016	*16·95*	$+\infty$	—
15·15	+ ·221	±·016			

This, it should be observed, is merely a statement of facts, and does not involve any assumption as to the distribution being really normal.

6. *Interpolation**. By means of these values of x, we can interpolate for values of X lying towards the extremities of the range, where the differences

* For a fuller discussion of the methods employed in this and the next two sections, see *Journal of the Royal Statistical Society*, Vol. LXIII. pp. 433—451.

of N_1 or N_2 are usually irregular. Thus, calculating the values of x for $X = 14{\cdot}05$ and $14{\cdot}15$ by means of the first difference alone, and thence calculating $\frac{1}{2}(1+\alpha)$ from Table I. (remembering that x and α are negative), we get the following results, as compared with the actual observations:—

X	x	$\frac{1}{2}(1-\alpha)$	$\frac{1}{2}(1+\alpha)$	$N_1=\frac{1}{2}N(1+\alpha)$	
				Calculated value	Actual value
14·05	−2·032	·9789	·0211	63	61
14·15	−1·829	·9663	·0337	101	97

It is quite possible that the discrepancy between the calculated and the actual values is mainly due to the errors of random selection of the 118 individuals lying between $X = 13{\cdot}95$ and $X = 14{\cdot}25$.

7. *Certain Special Cases.* The method is especially useful (*a*) where the differences in X are irregular or are large in comparison with the standard deviation, and (*b*) in dealing with the "arrays" in cases of normal or nearly normal correlation. As an example of the former, Prof. Pearson has provided me with the following results obtained by Miss C. D. Fawcett.

Mottling of Mimulus Luteus.

X=Number of splotches	Less than 50	50 to 61	62 to 71	More than 71	Total
Number of individuals	18	43	87	56	204

This gives three values of x in terms of X, viz:—

X	$49\frac{1}{2}$	$61\frac{1}{2}$	$71\frac{1}{2}$
x	$-1{\cdot}352 \pm {\cdot}084$	$-{\cdot}527 \pm {\cdot}062$	$+{\cdot}599 \pm {\cdot}063$

The differences in x are ·825 and 1·126, whereas the differences in X are in the ratio of 6 : 5. Having regard to the probable errors, it is very doubtful whether the distribution can be treated as normal. If it can be, the true values of x may be somewhat as follows:—

X	$49\frac{1}{2}$	$61\frac{1}{2}$	$71\frac{1}{2}$
x (corrected)	−1·503	−·417	+·488
Ratio of correction to P. E.	−1·8	+1·8	−1·8

These would give a mean of 66·61, and a standard deviation of 11·05, the probable errors being respectively ± ·52 and ± ·37.

This, of course, is guesswork. If there are more than three or four values of x to be dealt with, we can do our guessing by graphic methods. The values of x should be plotted as ordinates, with a mark on each side to shew the probable error: and we have then to draw a straight line which shall make the error as small as possible, allowance being made for the different values of the probable error in the different ordinates so plotted. It must be remembered that the errors are not independent, but correlated: the correlation between the errors in any pair of values of x being positive.

If we want to proceed more rigidly, we must use the actual values of x, and take account of differences. This is practically equivalent to regarding X as an unknown function of some other quantity Y, whose values are normally distributed; the relation between X and Y being such that dX/dY is always positive.

8. *Calculation of ordinates.* The ordinate Z of the curve of frequency is dN_1/dX. In most cases this can be calculated directly from the data, by the first formula of (5) (taking $u=N_1$) or a similar formula. Where the differences in u are too irregular for this, but the differences in x are comparatively regular, we can use the data in the form shewn in § 5: we have then

$$Z=N\,.\,d\tfrac{1}{2}(1+a)/dx\,.\;dx/dX=Nz\,.\,dx/dX.$$

The values of dx/dX are given by the data, and the values of z are found from Table II.

9. *Testing for normal distribution.* This is the purpose for which tables such as Table I. are most frequently employed. It is not necessary to give any examples here.

10. *Calculation of correlation-volumes.* The formula, given by Professor Pearson*, for calculating the double-integral expressing normal correlation, involves a factor

$$\frac{1}{2\pi}\,e^{-\frac{1}{2}(x^2+y^2)},$$

which is the product of the values of z as given by Table II. for x and for y respectively, and therefore is easily found from that Table.

Construction of the Tables.

11. The tabulated values are all given to seven decimal places (and, in the latter part of Tables I. and II., to ten decimal places). They were originally calculated to two or three more places, the final figures being then corrected. Where the final figure was doubtful, the value was calculated specially. The differences are taken from the larger table, the last figure being corrected: but doubtful values were not specially calculated.

For constructing Tables I. and II. up to $x=2{\cdot}50$, the values of z were found for the intermediate values ·005, ·015, ·025... of x by successive multiplication; each tenth value being checked by Newman's table† of $e^{-\theta}$. The differences being then taken, the values of $\frac{1}{2}(1+a)$ for

* *Phil. Trans.* series A, Vol. 195, pp. 1—47.

† *Camb. Phil. Soc. Trans.* Vol. XIII. Pt. 3.

$x = \cdot 01, \cdot 02 \ldots$ were obtained by quadrature*, and those of z by interpolation. For the remainder of the two tables, the values of z were found from those of $\log_{10} z$, which are easily calculated, and thence the values of $\frac{1}{2}(1+\alpha)$ were obtained by quadrature. For checking the table of $\frac{1}{2}(1+\alpha)$, values were directly calculated at intervals†: and both tables were further checked by the calculations required where a final figure was doubtful.

For constructing Tables III. and IV., the values were first obtained approximately to seven places: and the tables were then extended, by a method explained elsewhere‡. The extension gave z to nine and x to eleven places. The table was checked by direct calculation for $\alpha = \cdot 1, \cdot 2, \cdot 3 \ldots$.

* See *Proc. of Lond. Math. Soc.* Vol. XXXI. pp. 479—482.

† Some use was also made of Burgess's tables (*Trans. Roy. Soc. Edin.*, Vol. XXXIX., Pt. 2, No. 9), in which α is given (to a large number of figures) in terms of $t \equiv x/\sqrt{2}$. But they were only used incidentally and the two sets of tables may be regarded as independently calculated.

‡ *Proc. of Lond. Math. Soc.* Vol. XXXI. pp. 423, 439.

TABLES I. AND II. *Area and Ordinate in terms of Abscissa.*

Note.

For values of the abscissa x from $\cdot 00$ to $4 \cdot 50$, the values of the area $\frac{1}{2}(1+\alpha)$ and of the ordinate z are given to 7 decimal places (pp. 182–7). For values of x from $4 \cdot 50$ to $6 \cdot 00$, the values of $\frac{1}{2}(1+\alpha)$ and of z are given to ten decimal places (p. 188), but the initial figures are omitted. Hence, in using this latter portion of the tables the figures in the column for $\frac{1}{2}(1+\alpha)$ must have $\cdot 99999$ prefixed, and those in the column for z must have sufficient zeros prefixed to bring up the total of decimal figures to ten. For example, against $x = 5 \cdot 75$ we have 99955 and 264, but we must read $\frac{1}{2}(1+\alpha) = \cdot 9999999955$ and $z = \cdot 0000000264$.

TABLES I. AND II. *Area and Ordinate in terms of Abscissa.*

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −	z	Δ −	Δ^2 −
·00	·5000000		0	·3989423		399
·01	·5039894	39894	4	·3989223	199	399
·02	·5079783	39890	8	·3988625	598	399
·03	·5119665	39882	12	·3987628	997	398
·04	·5159534	39870	16	·3986233	1395	398
·05	·5199388	39854	20	·3984439	1793	397
		39834			2191	
·06	·5239222		24	·3982248		397
·07	·5279032	39810	28	·3979661	2588	396
·08	·5318814	39782	32	·3976677	2984	395
·09	·5358564	39750	36	·3973298	3379	394
·10	·5398278	39714	40	·3969525	3773	393
		39675			4166	
·11	·5437953		44	·3965360		392
·12	·5477584	39631	48	·3960802	4558	390
·13	·5517168	39584	51	·3955854	4948	389
·14	·5556700	39532	55	·3950517	5337	387
·15	·5596177	39477	59	·3944793	5724	386
		39418			6110	
·16	·5635595		63	·3938684		384
·17	·5674949	39355	67	·3932190	6493	382
·18	·5714237	39288	71	·3925315	6875	380
·19	·5753454	39217	74	·3918060	7255	378
·20	·5792597	39143	78	·3910427	7633	375
		39065			8008	
·21	·5831662		82	·3902419		373
·22	·5870644	38983	86	·3894038	8381	371
·23	·5909541	38897	89	·3885286	8752	368
·24	·5948349	38808	93	·3876166	9120	365
·25	·5987063	38715	97	·3866681	9485	362
		38618			9847	
·26	·6025681		100	·3856834		360
·27	·6064199	38518	104	·3846627	10207	357
·28	·6102612	38414	107	·3836063	10564	354
·29	·6140919	38306	111	·3825146	10917	350
·30	·6179114	38195	114	·3813878	11268	347
		38081			11615	
·31	·6217195		118	·3802264		344
·32	·6255158	37963	121	·3790305	11958	340
·33	·6293000	37842	125	·3778007	12298	337
·34	·6330717	37717	128	·3765372	12635	333
·35	·6368307	37589	131	·3752403	12968	329
		37458			13297	
·36	·6405764		135	·3739106		325
·37	·6443088	37323	138	·3725483	13623	322
·38	·6480273	37185	141	·3711539	13944	318
·39	·6517317	37044	144	·3697277	14262	313
·40	·6554217	36900	147	·3682701	14575	309
		36753			14885	
·41	·6590970		150	·3667817		305
·42	·6627573	36602	153	·3652627	15190	301
·43	·6664022	36449	156	·3637136	15491	296
·44	·6700314	36293	159	·3621349	15787	292
·45	·6736448	36133	162	·3605270	16079	288
		35971			16367	
·46	·6772419		165	·3588903		283
·47	·6808225	35806	168	·3572253	16650	278
·48	·6843863	35638	171	·3555325	16928	274
·49	·6879331	35467	173	·3538124	17202	269
·50	·6914625	35294	176	·3520653	17470	264

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −
·50	·6914625		176
·51	·6949743	35118	179
·52	·6984682	34939	181
·53	·7019440	34758	184
·54	·7054015	34574	186
·55	·7088403	34388	189
		34200	
·56	·7122603		191
·57	·7156612	34009	193
·58	·7190427	33815	196
·59	·7224047	33620	198
·60	·7257469	33422	200
		33222	
·61	·7290691		202
·62	·7323711	33020	204
·63	·7356527	32816	206
·64	·7389137	32610	208
·65	·7421539	32402	210
		32192	
·66	·7453731		212
·67	·7485711	31980	214
·68	·7517478	31767	215
·69	·7549029	31551	217
·70	·7580363	31334	219
		31116	
·71	·7611479		220
·72	·7642375	30896	222
·73	·7673049	30674	223
·74	·7703500	30451	225
·75	·7733726	30226	226
		30001	
·76	·7763727		227
·77	·7793501	29773	228
·78	·7823046	29545	230
·79	·7852361	29316	231
·80	·7881446	29085	232
		28853	
·81	·7910299		233
·82	·7938919	28620	234
·83	·7967306	28387	235
·84	·7995458	28152	235
·85	·8023375	27917	236
		27680	
·86	·8051055		237
·87	·8078498	27443	238
·88	·8105703	27205	238
·89	·8132671	26967	239
·90	·8159399	26728	239
		26489	
·91	·8185887		240
·92	·8212136	26249	240
·93	·8238145	26008	241
·94	·8263912	25768	241
·95	·8289439	25527	241
		25285	
·96	·8314724		242
·97	·8339768	25044	242
·98	·8364569	24802	242
·99	·8389129	24560	242
1·00	·8413447	24318	242

TABLES I. AND II.—(*continued*).

z	Δ −	Δ^2 −
·3520653		
	17734	264
·3502919		
	17994	259
·3484925		
	18248	254
·3466677		
	18497	249
·3448180		
	18741	244
·3429439		
	18981	239
·3410458		
	19215	234
·3391243		
	19444	229
·3371799		
	19667	224
·3352132		
	19886	219
·3332246		
	20099	213
·3312147		
	20307	208
·3291840		
	20510	203
·3271330		
	20707	197
·3250623		
	20899	192
·3229724		
	21086	187
·3208638		
	21267	181
·3187371		
	21442	176
·3165929		
	21613	170
·3144317		
	21777	165
·3122539		
	21936	159
·3100603		
	22090	154
·3078513		
	22239	148
·3056274		
	22381	143
·3033893		
	22519	137
·3011374		
	22650	132
·2988724		
	22777	126
·2965948		
	22897	121
·2943050		
	23013	115
·2920038		
	23122	110
·2896916		
	23227	104
·2873689		
	23325	99
·2850364		
	23419	93
·2826945		
	23507	88
·2803438		
	23589	83
·2779849		
	23666	77
·2756182		
	23738	72
·2732444		
	23805	66
·2708640		
	23866	61
·2684774		
	23922	56
·2660852		
	23972	51
·2636880		
	24017	45
·2612863		
	24058	40
·2588805		
	24093	35
·2564713		
	24122	30
·2540591		
	24147	25
·2516443		
	24167	20
·2492277		
	24182	15
·2468095		
	24191	10
·2443904		
	24196	5
·2419707		0

x °	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −	z	Δ −	Δ^2 +
1·00	·8413447			·2419707		
		24076	242		24196	0
1·01	·8437524			·2395511		
		23834	242		24191	5
1·02	·8461358			·2371320		
		23592	242		24182	10
1·03	·8484950			·2347138		
		23351	242		24168	14
1·04	·8508300			·2322970		
		23109	242		24149	19
1·05	·8531409			·2298821		
		22868	241		24125	24
1·06	·8554277			·2274696		
		22626	241		24097	28
1·07	·8576903			·2250599		
		22386	240		24064	33
1·08	·8599289			·2226535		
		22145	240		24027	37
1·09	·8621434			·2202508		
		21905	240		23986	41
1·10	·8643339			·2178522		
		21665	240		23940	46
1·11	·8665005			·2154582		
		21426	239		23890	50
1·12	·8686431			·2130691		
		21188	239		23836	54
1·13	·8707619			·2106856		
		20950	238		23778	58
1·14	·8728568			·2083078		
		20712	237		23715	62
1·15	·8749281			·2059363		
		20475	237		23649	66
1·16	·8769756			·2035714		
		20239	236		23578	70
1·17	·8789995			·2012135		
		20004	235		23504	74
1·18	·8809999			·1988631		
		19769	235		23426	78
1·19	·8829768			·1965205		
		19535	234		23344	82
1·20	·8849303			·1941861		
		19302	233		23259	85
1·21	·8868606			·1918602		
		19070	232		23170	89
1·22	·8887676			·1895432		
		18839	231		23077	93
1·23	·8906514			·1872354		
		18609	230		22981	96
1·24	·8925123			·1849373		
		18379	229		22882	99
1·25	·8943502			·1826491		
		18151	228		22779	103
1·26	·8961653			·1803712		
		17924	227		22673	106
1·27	·8979577			·1781038		
		17697	226		22564	109
1·28	·8997274			·1758474		
		17472	225		22452	112
1·29	·9014747			·1736022		
		17248	224		22337	115
1·30	·9031995			·1713686		
		17026	223		22218	118
1·31	·9049021			·1691468		
		16804	222		22097	121
1·32	·9065825			·1669370		
		16584	220		21973	124
1·33	·9082409			·1647397		
		16365	219		21847	127
1·34	·9098773			·1625551		
		16147	218		21717	129
1·35	·9114920			·1603833		
		15930	217		21585	132
1·36	·9130850			·1582248		
		15715	215		21451	134
1·37	·9146565			·1560797		
		15501	214		21314	137
1·38	·9162067			·1539483		
		15289	212		21175	139
1·39	·9177356			·1518308		
		15078	211		21033	142
1·40	·9192433			·1497275		
		14868	210		20890	144
1·41	·9207302			·1476385		
		14660	208		20744	146
1·42	·9221962			·1455641		
		14453	207		20596	148
1·43	·9236415			·1435046		
		14248	205		20446	150
1·44	·9250663			·1414600		
		14044	204		20294	152
1·45	·9264707			·1394306		
		13842	202		20140	154
1·46	·9278550			·1374165		
		13642	201		19985	155
1·47	·9292191			·1354181		
		13443	199		19828	157
1·48	·9305634			·1334353		
		13245	197		19669	159
1·49	·9318879			·1314684		
		13049	196		19508	160
1·50	·9331928		194	·1295176		162

TABLES I. AND II.—(*continued*).

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −	z	Δ −	Δ^2 +
1·50	·9331928	12855	194	·1295176	19346	162
1·51	·9344783	12662	193	·1275830	19183	163
1·52	·9357445	12471	191	·1256646	19018	165
1·53	·9369916	12282	189	·1237628	18853	166
1·54	·9382198	12094	188	·1218775	18685	167
1·55	·9394292	11908	186	·1200090	18517	168
1·56	·9406201	11724	184	·1181573	18348	169
1·57	·9417924	11541	183	·1163225	18177	170
1·58	·9429466	11360	181	·1145048	18006	171
1·59	·9440826	11181	179	·1127042	17834	172
1·60	·9452007	11004	177	·1109208	17661	173
1·61	·9463011	10828	176	·1091548	17487	174
1·62	·9473839	10654	174	·1074061	17312	174
1·63	·9484493	10482	172	·1056748	17137	175
1·64	·9494974	10311	170	·1039611	16962	176
1·65	·9505285	10142	169	·1022649	16786	176
1·66	·9515428	9975	167	·1005864	16609	177
1·67	·9525403	9810	165	·0989255	16432	177
1·68	·9535213	9647	163	·0972823	16255	177
1·69	·9544860	9485	162	·0956568	16077	178
1·70	·9554345	9325	160	·0940491	15899	178
1·71	·9563671	9167	158	·0924591	15722	178
1·72	·9572838	9011	156	·0908870	15544	178
1·73	·9581849	8856	155	·0893326	15366	178
1·74	·9590705	8704	153	·0877961	15188	178
1·75	·9599408	8553	151	·0862773	15010	178
1·76	·9607961	8403	149	·0847764	14832	178
1·77	·9616364	8256	147	·0832932	14654	178
1·78	·9624620	8110	146	·0818278	14477	177
1·79	·9632730	7966	144	·0803801	14300	177
1·80	·9640697	7824	142	·0789502	14123	177
1·81	·9648521	7684	140	·0775379	13946	176
1·82	·9656205	7545	139	·0761433	13770	176
1·83	·9663750	7409	137	·0747663	13594	176
1·84	·9671159	7273	135	·0734068	13419	175
1·85	·9678432	7140	133	·0720649	13245	175
1·86	·9685572	7009	132	·0707404	13071	174
1·87	·9692581	6879	130	·0694333	12897	173
1·88	·9699460	6751	128	·0681436	12725	173
1·89	·9706210	6624	126	·0668711	12553	172
1·90	·9712834	6500	125	·0656158	12382	171
1·91	·9719334	6377	123	·0643777	12211	170
1·92	·9725711	6255	121	·0631566	12041	170
1·93	·9731966	6136	120	·0619524	11873	169
1·94	·9738102	6018	118	·0607652	11705	168
1·95	·9744119	5902	116	·0595947	11538	167
1·96	·9750021	5787	115	·0584409	11372	166
1·97	·9755808	5674	113	·0573038	11206	165
1·98	·9761482	5563	111	·0561831	11042	164
1·99	·9767045	5453	110	·0550789	10879	163
2·00	·9772499		108	·0539910		162

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −
2·00	·9772499	5345	108
2·01	·9777844	5239	106
2·02	·9783083	5134	105
2·03	·9788217	5031	103
2·04	·9793248	4929	102
2·05	·9798178	4829	100
2·06	·9803007	4731	98
2·07	·9807738	4634	97
2·08	·9812372	4539	95
2·09	·9816911	4445	94
2·10	·9821356	4352	92
2·11	·9825708	4262	91
2·12	·9829970	4172	89
2·13	·9834142	4084	88
2·14	·9838226	3998	86
2·15	·9842224	3913	85
2·16	·9846137	3829	84
2·17	·9849966	3747	82
2·18	·9853713	3666	81
2·19	·9857379	3587	79
2·20	·9860966	3509	78
2·21	·9864474	3432	77
2·22	·9867906	3357	75
2·23	·9871263	3283	74
2·24	·9874545	3210	73
2·25	·9877755	3138	71
2·26	·9880894	3068	70
2·27	·9883962	2999	69
2·28	·9886962	2932	68
2·29	·9889893	2865	66
2·30	·9892759	2800	65
2·31	·9895559	2736	64
2·32	·9898296	2674	63
2·33	·9900969	2612	62
2·34	·9903581	2552	60
2·35	·9906133	2492	59
2·36	·9908625	2434	58
2·37	·9911060	2377	57
2·38	·9913437	2321	56
2·39	·9915758	2267	55
2·40	·9918025	2213	54
2·41	·9920237	2160	53
2·42	·9922397	2108	52
2·43	·9924506	2058	51
2·44	·9926564	2008	50
2·45	·9928572	1960	49
2·46	·9930531	1912	48
2·47	·9932443	1865	47
2·48	·9934309	1820	46
2·49	·9936128	1775	45
2·50	·9937903		44

TABLES I. AND II.—(*continued*).

z	Δ −	Δ^2 +
·0539910		
	10717	
·0529192		162
	10557	161
·0518636		
	10397	160
·0508239		
	10238	159
·0498001		
	10081	157
·0487920		
	9924	156
·0477996		
	9769	155
·0468226		
	9616	154
·0458611		
	9463	153
·0449148		
	9312	151
·0439836		
	9162	150
·0430674		
	9013	149
·0421661		
	8866	147
·0412795		
	8720	146
·0404076		
	8575	145
·0395500		
	8432	143
·0387069		
	8290	142
·0378779		
	8149	140
·0370629		
	8010	139
·0362619		
	7873	138
·0354746		
	7737	136
·0347009		
	7602	135
·0339408		
	7468	133
·0331939		
	7337	132
·0324603		
	7206	130
·0317397		
	7077	129
·0310319		
	6950	127
·0303370		
	6824	126
·0296546		
	6699	125
·0289847		
	6576	123
·0283270		
	6455	122
·0276816		
	6335	120
·0270481		
	6216	119
·0264265		
	6099	117
·0258166		
	5984	116
·0252182		
	5870	114
·0246313		
	5757	113
·0240556		
	5646	111
·0234910		
	5536	110
·0229374		
	5428	108
·0223945		
	5322	107
·0218624		
	5217	105
·0213407		
	5113	104
·0208294		
	5011	102
·0203284		
	4910	101
·0198374		
	4811	99
·0193563		
	4713	98
·0188850		
	4617	96
·0184233		
	4522	95
·0179711		
	4428	93
·0175283		92

x	$\frac{1}{2}(1+\alpha)$	Δ +	Δ^2 −
2·50	·9937903		44
		1731	
2·51	·9939634		43
		1688	
2·52	·9941323		42
		1646	
2·53	·9942969		41
		1605	
2·54	·9944574		40
		1565	
2·55	·9946139		39
		1525	
2·56	·9947664		39
		1487	
2·57	·9949151		38
		1449	
2·58	·9950600		37
		1412	
2·59	·9952012		36
		1376	
2·60	·9953388		35
		1341	
2·61	·9954729		35
		1306	
2·62	·9956035		34
		1272	
2·63	·9957308		33
		1239	
2·64	·9958547		32
		1207	
2·65	·9959754		32
		1176	
2·66	·9960930		31
		1145	
2·67	·9962074		30
		1115	
2·68	·9963189		29
		1085	
2·69	·9964274		29
		1056	
2·70	·9965330		28
		1028	
2·71	·9966358		27
		1001	
2·72	·9967359		27
		974	
2·73	·9968333		26
		948	
2·74	·9969280		26
		922	
2·75	·9970202		25
		897	
2·76	·9971099		24
		873	
2·77	·9971972		24
		849	
2·78	·9972821		23
		825	
2·79	·9973646		23
		803	
2·80	·9974449		22
		781	
2·81	·9975229		22
		759	
2·82	·9975988		21
		738	
2·83	·9976726		21
		717	
2·84	·9977443		20
		697	
2·85	·9978140		20
		678	
2·86	·9978818		19
		658	
2·87	·9979476		19
		640	
2·88	·9980116		18
		622	
2·89	·9980738		18
		604	
2·90	·9981342		17
		587	
2·91	·9981929		17
		570	
2·92	·9982498		16
		553	
2·93	·9983052		16
		537	
2·94	·9983589		16
		522	
2·95	·9984111		15
		507	
2·96	·9984618		15
		492	
2·97	·9985110		14
		478	
2·98	·9985588		14
		464	
2·99	·9986051		14
		450	
3·00	·9986501		13

z	Δ −	Δ^2 +
·0175283		92
	4336	
·0170947		91
	4246	
·0166701		89
	4157	
·0162545		88
	4069	
·0158476		86
	3982	
·0154493		85
	3897	
·0150596		84
	3814	
·0146782		82
	3731	
·0143051		81
	3650	
·0139401		80
	3571	
·0135830		78
	3493	
·0132337		77
	3416	
·0128921		76
	3340	
·0125581		74
	3266	
·0122315		73
	3193	
·0119122		72
	3121	
·0116001		70
	3051	
·0112951		69
	2981	
·0109969		68
	2913	
·0107056		67
	2847	
·0104209		66
	2781	
·0101428		64
	2717	
·0098712		63
	2654	
·0096058		62
	2592	
·0093466		61
	2531	
·0090936		60
	2471	
·0088465		59
	2413	
·0086052		57
	2355	
·0083697		56
	2299	
·0081398		55
	2244	
·0079155		54
	2189	
·0076965		53
	2136	
·0074829		52
	2084	
·0072744		51
	2033	
·0070711		50
	1983	
·0068728		49
	1934	
·0066793		48
	1886	
·0064907		47
	1839	
·0063067		46
	1793	
·0061274		45
	1748	
·0059525		44
	1704	
·0057821		43
	1661	
·0056160		42
	1619	
·0054541		41
	1578	
·0052963		40
	1537	
·0051426		40
	1497	
·0049929		39
	1459	
·0048470		38
	1421	
·0047050		37
	1384	
·0045666		36
	1347	
·0044318		35

TABLES I. AND II.—(*continued*).

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −	z	Δ −	Δ^2 +
3·00	·9986501	437	13	·0044318	1312	35
3·01	·9986938	424	13	·0043007	1277	35
3·02	·9987361	411	13	·0041729	1243	34
3·03	·9987772	399	12	·0040486	1210	33
3·04	·9988171	387	12	·0039276	1178	32
3·05	·9988558	375	12	·0038098	1146	32
3·06	·9988933	364	11	·0036951	1115	31
3·07	·9989297	353	11	·0035836	1085	30
3·08	·9989650	342	11	·0034751	1056	29
3·09	·9989992	332	10	·0033695	1027	29
3·10	·9990324	322	10	·0032668	999	28
3·11	·9990646	312	10	·0031669	971	27
3·12	·9990957	302	10	·0030698	944	27
3·13	·9991260	293	9	·0029754	918	26
3·14	·9991553	284	9	·0028835	893	26
3·15	·9991836	275	9	·0027943	868	25
3·16	·9992112	267	9	·0027075	843	24
3·17	·9992378	258	8	·0026231	820	24
3·18	·9992636	250	8	·0025412	797	23
3·19	·9992886	242	8	·0024615	774	23
3·20	·9993129	235	8	·0023841	752	22
3·21	·9993363	227	7	·0023089	731	21
3·22	·9993590	220	7	·0022358	710	21
3·23	·9993810	213	7	·0021649	689	20
3·24	·9994024	206	7	·0020960	669	20
3·25	·9994230	200	7	·0020290	650	19
3·26	·9994429	193	6	·0019641	631	19
3·27	·9994623	187	6	·0019010	612	18
3·28	·9994810	181	6	·0018397	595	18
3·29	·9994991	175	6	·0017803	577	17
3·30	·9995166	169	6	·0017226	560	17
2·31	·9995335	164	6	·0016666	543	17
3·32	·9995499	159	5	·0016122	527	16
3·33	·9995658	153	5	·0015595	512	16
3·34	·9995811	148	5	·0015084	496	15
3·35	·9995959	143	5	·0014587	481	15
3·36	·9996103	139	5	·0014106	467	15
3·37	·9996242	134	5	·0013639	453	14
3·38	·9996376	130	4	·0013187	439	14
3·39	·9996505	125	4	·0012748	426	13
3·40	·9996631	121	4	·0012322	413	13
3·41	·9996752	117	4	·0011910	400	13
3·42	·9996869	113	4	·0011510	388	12
3·43	·9996982	109	4	·0011122	376	12
3·44	·9997091	106	4	·0010747	364	12
3·45	·9997197	102	4	·0010383	353	11
3·46	·9997299	99	3	·0010030	342	11
3·47	·9997398	95	3	·0009689	331	11
3·48	·9997493	92	3	·0009358	320	10
3·49	·9997585	89	3	·0009037	310	10
3·50	·9997674		3	·0008727		10

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −
3·50	·9997674	86	3
3·51	·9997759	83	3
3·52	·9997842	80	3
3·53	·9997922	77	3
3·54	·9997999	74	3
3·55	·9998074	72	3
3·56	·9998146	69	3
3·57	·9998215	67	2
3·58	·9998282	65	2
3·59	·9998347	62	2
3·60	9998409	60	2
3·61	9998469	58	2
3·62	·9998527	56	2
3·63	·9998583	54	2
3·64	·9998637	52	2
3·65	·9998689	50	2
3·66	·9998739	48	2
3·67	·9998787	47	2
3·68	·9998834	45	2
3·69	·9998879	43	2
3·70	·9998922	42	2
3·71	·9998964	40	2
3·72	·9999004	39	1
3·73	·9999043	37	1
3·74	·9999080	36	1
3·75	·9999116	35	1
3·76	·9999150	33	1
3·77	·9999184	32	1
3·78	·9999216	31	1
3·79	·9999247	30	1
3·80	·9999277	29	1
3·81	·9999305	28	1
3·82	·9999333	27	1
3·83	·9999359	26	1
3·84	·9999385	25	1
3·85	·9999409	24	1
3·86	·9999433	23	1
3·87	·9999456	22	1
3·88	·9999478	21	1
3·89	·9999499	20	1
3·90	·9999519	19	1
3·91	·9999539	19	1
3·92	·9999557	18	1
3·93	·9999575	17	1
3·94	·9999593	17	1
3·95	·9999609	16	1
3·96	·9999625	15	1
3·97	·9999641	15	1
3·98	·9999655	14	1
3·99	·9999670	14	1
4·00	·9999683		1

TABLES I. AND II.—(*continued*).

z	Δ −	Δ^2 +
·0008727		
	301	10
·0008426		
	291	10
·0008135		
	282	9
·0007853		
	273	9
·0007581		
	264	8
·0007317		
	256	
·0007061		8
	247	
·0006814		8
	239	
·0006575		8
	232	
·0006343		8
	224	
·0006119		7
	217	
·0005902		7
	210	
·0005693		7
	203	
·0005490		7
	196	
·0005294		6
	189	
·0005105		6
	183	
·0004921		6
	177	
·0004744		6
	171	
·0004573		6
	165	
·0004408		6
	160	
·0004248		5
	155	
·0004093		5
	149	
·0003944		5
	144	
·0003800		5
	139	
·0003661		5
	135	
·0003526		5
	130	
·0003396		4
	125	
·0003271		4
	121	
·0003149		4
	117	
·0003032		4
	113	
·0002919		4
	109	
·0002810		4
	105	
·0002705		4
	102	
·0002604		4
	98	
·0002506		3
	95	
·0002411		3
	91	
·0002320		3
	88	
·0002232		3
	85	
·0002147		3
	82	
·0002065		3
	79	
·0001987		3
	76	
·0001910		3
	73	
·0001837		3
	71	
·0001766		3
	68	
·0001698		2
	66	
·0001633		2
	63	
·0001569		2
	61	
·0001508		2
	59	
·0001449		2
	57	
·0001393		2
	55	
·0001338		2

x	$\frac{1}{2}(1+a)$	Δ +	Δ^2 −
4·00	·9999683		
		13	1
4·01	·9999696		
		13	1
4·02	·9999709		
		12	0
4·03	·9999721		
		12	
4·04	·9999733		
		11	
4·05	·9999744		
		11	
4·06	·9999755		
		10	
4·07	·9999765		
		10	
4·08	·9999775		
		9	
4·09	·9999784		
		9	
4·10	·9999793		
		9	
4·11	·9999802		
		8	
4·12	·9999811		
		8	
4·13	·9999819		
		8	
4·14	·9999826		
		7	
4·15	·9999834		
		7	
4·16	·9999841		
		7	
4·17	·9999848		
		7	
4·18	·9999854		
		6	
4·19	·9999861		
		6	
4·20	·9999867		
		6	
4·21	·9999872		
		6	
4·22	·9999878		
		5	
4·23	·9999883		
		5	
4·24	·9999888		
		5	
4·25	·9999893		
		5	
4·26	·9999898		
		4	
4·27	·9999902		
		4	
4·28	·9999907		
		4	
4·29	·9999911		
		4	
4·30	·9999915		
		4	
4·31	·9999918		
		4	
4·32	·9999922		
		3	
4·33	·9999925		
		3	
4·34	·9999929		
		3	
4·35	·9999932		
		3	
4·36	·9999935		
		3	
4·37	·9999938		
		3	
4·38	·9999941		
		3	
4·39	·9999943		
		3	
4·40	·9999946		
		2	
4·41	·9999948		
		2	
4·42	·9999951		
		2	
4·43	·9999953		
		2	
4·44	·9999955		
		2	
4·45	·9999957		
		2	
4·46	·9999959		
		2	
4·47	·9999961		
		2	
4·48	·9999963		
		2	
4·49	·9999964		
		2	
4·50	·9999966		

z	Δ −	Δ^2 +
·0001338		
	53	2
·0001286		
	51	2
·0001235		
	49	2
·0001186		
	47	2
·0001140		
	45	2
·0001094		
	43	
·0001051		2
	42	
·0001009		2
	40	
·0000969		2
	39	
·0000930		1
	37	
·0000893		1
	36	
·0000857		1
	35	
·0000822		1
	33	
·0000789		1
	32	
·0000757		1
	31	
·0000726		1
	30	
·0000697		1
	28	
·0000668		1
	27	
·0000641		1
	26	
·0000615		1
	25	
·0000589		1
	24	
·0000565		1
	23	
·0000542		1
	22	
·0000519		1
	22	
·0000498		1
	21	
·0000477		1
	20	
·0000457		1
	19	
·0000438		1
	18	
·0000420		1
	18	
·0000402		1
	17	
·0000385		1
	16	
·0000369		1
	16	
·0000354		1
	15	
·0000339		1
	14	
·0000324		1
	14	
·0000310		1
	13	
·0000297		1
	13	
·0000284		1
	12	
·0000272		0
	12	
·0000261		
	11	
·0000249		
	11	
·0000239		
	10	
·0000228		
	10	
·0000218		
	9	
·0000209		
	9	
·0000200		
	9	
·0000191		
	8	
·0000183		
	8	
·0000175		
	8	
·0000167		
	7	
·0000160		

TABLES I. AND II.—(*continued*). See Note, p. 181.

x	$\frac{1}{2}(1+a)$	z
4·50	66023	159837
4·51	67586	152797
4·52	69080	146051
4·53	70508	139590
4·54	71873	133401
4·55	73177	127473
4·56	74423	121797
4·57	75614	116362
4·58	76751	111159
4·59	77838	106177
4·60	78875	101409
4·61	79867	96845
4·62	80813	92477
4·63	81717	88297
4·64	82580	84298
4·65	83403	80472
4·66	84190	76812
4·67	84940	73311
4·68	85656	69962
4·69	86340	66760
4·70	86992	63698
4·71	87614	60771
4·72	88208	57972
4·73	88774	55296
7·74	89314	52739
4·75	89829	50295
4·76	90320	47960
4·77	90789	45728
4·78	91235	43596
4·79	91661	41559
4·80	92067	39613
4·81	92453	37755
4·82	92822	35980
4·83	93173	34285
4·84	93508	32667
4·85	93827	31122
4·86	94131	29647
4·87	94420	28239
4·88	94696	26895
4·89	94958	25613
4·90	95208	24390
4·91	95446	23222
4·92	95673	22108
4·93	95889	21046
4·94	96094	20033
4·95	96289	19066
4·96	96475	18144
4·97	96652	17265
4·98	96821	16428
4·99	96981	15629

x	$\frac{1}{2}(1+a)$	z
5·00	97133	14867
5·01	97278	14141
5·02	97416	13450
5·03	97548	12791
5·04	97672	12162
5·05	97791	11564
5·06	97904	10994
5·07	98011	10451
5·08	98113	9934
5·09	98210	9441
5·10	98302	8972
5·11	98389	8526
5·12	98472	8101
5·13	98551	7696
5·14	98626	7311
5·15	98698	6944
5·16	98765	6595
5·17	98830	6263
5·18	98891	5947
5·19	98949	5647
5·20	99004	5361
5·21	99056	5089
5·22	99105	4831
5·23	99152	4585
5·24	99197	4351
5·25	99240	4128
5·26	99280	3917
5·27	99318	3716
5·28	99354	3525
5·29	99388	3344
5·30	99421	3171
5·31	99452	3007
5·32	99481	2852
5·33	99509	2704
5·34	99535	2563
5·35	99560	2430
5·36	99584	2303
5·37	99606	2183
5·38	99628	2069
5·39	99648	1960
5·40	99667	1857
5·41	99685	1760
5·42	99702	1667
5·43	99718	1579
5·44	99734	1495
5·45	99748	1416
5·46	99762	1341
5·47	99775	1270
5·48	99787	1202
5·49	99799	1138

x	$\frac{1}{2}(1+a)$	z
5·50	99810	1077
5·51	99821	1019
5·52	99831	965
5·53	99840	913
5·54	99849	864
5·55	99857	817
5·56	99865	773
5·57	99873	731
5·58	99880	691
5·59	99886	654
5·60	99893	618
5·61	99899	585
5·62	99905	553
5·63	99910	522
5·64	99915	494
5·65	99920	467
5·66	99924	441
5·67	99929	417
5·68	99933	394
5·69	99936	372
5·70	99940	351
5·71	99944	332
5·72	99947	313
5·73	99950	296
5·74	99953	280
5·75	99955	264
5·76	99958	249
5·77	99960	235
5·78	99963	222
5·79	99965	210
5·80	99967	198
5·81	99969	187
5·82	99971	176
5·83	99972	166
5·84	99974	157
5·85	99975	148
5·86	99977	139
5·87	99978	131
5·88	99979	124
5·89	99981	117
5·90	99982	110
5·91	99983	104
5·92	99984	98
5·93	99985	92
5·94	99986	87
5·95	99987	82
5·96	99987	77
5·97	99988	73
5·98	99989	68
5·99	99990	65
6·00	99990	61

TABLES III. AND IV. *Abscissa and Ordinate in terms of difference of Areas.*

a	x	Δ^* +	Δ^2 +	Δ^3 +	z	Δ −	Δ^2 −	Δ^3 −
·00	·0000000				·3989423			
		125335	0	20		313	627	0
·01	·0125335				·3989109			
		125354	20	20		940	627	0
·02	·0250689				·3988169			
		125394	39	20		1567	627	0
·03	·0376083				·3986603			
		125453	59	20		2194	627	0
·04	·0501536				·3984408			
		125532	79	20		2821	627	0
·05	·0627068				·3981587			
		125631	99	20		3449	628	1
·06	·0752699				·3978138			
		125750	119	20		4078	628	1
·07	·0878448				·3974060			
		125889	139	20		4707	629	1
·08	·1004337				·3969353			
		126048	159	20		5337	630	1
·09	·1130385				·3964016			
		126228	180	21		5967	631	1
·10	·1256613				·3958049			
		126429	201	21		6599	632	1
·11	·1383042				·3951450			
		126650	221	21		7232	633	1
·12	·1509692				·3944218			
		126893	243	21		7866	634	1
·13	·1636585				·3936352			
		127157	264	22		8501	635	1
·14	·1763742				·3927852			
		127443	286	22		9137	636	1
·15	·1891184				·3918715			
		127751	308	22		9775	638	2
·16	·2018935				·3908939			
		128081	330	23		10415	640	2
·17	·2147016				·3898525			
		128434	353	23		11056	641	2
·18	·2275450				·3887469			
		128811	376	24		11699	643	2
·19	·2404260				·3875769			
		129211	400	24		12344	645	2
·20	·2533471				·3863425			
		129635	424	25		12991	647	2
·21	·2663106				·3850434			
		130084	449	25		13641	649	2
·22	·2793190				·3836794			
		130559	474	26		14292	652	2
·23	·2923749				·3822501			
		131059	500	27		14946	654	3
·24	·3054808				·3807555			
		131586	527	27		15603	657	3
·25	·3186394				·3791952			
		132140	554	28		16262	659	3
·26	·3318533				·3775690			
		132722	582	29		16924	662	3
·27	·3451255				·3758766			
		133333	611	30		17589	665	3
·28	·3584588				·3741177			
		133973	640	30		18258	668	3
·29	·3718561				·3722919			
		134644	671	31		18929	672	3
·30	·3853205				·3703990			
		135346	702	32		19604	675	4
·31	·3988551				·3684386			
		136081	735	34		20283	679	4
·32	·4124631				·3664103			
		136849	768	35		20965	682	4
·33	·4261480				·3643138			
		137652	803	36		21651	686	4
·34	·4399132				·3621487			
		138490	839	37		22342	690	4
·35	·4537622				·3599146			
		139366	876	39		23036	695	4
·36	·4676988				·3576109			
		140281	914	40		23735	699	5
·37	·4817268				·3552374			
		141235	954	42		24439	704	5
·38	·4958503				·3527935			
		142231	996	43		25148	709	5
·39	·5100735				·3502788			
		143271	1039	45		25861	714	5
·40	·5244005				·3476926			
		144355	1085	47		26580	719	6
·41	·5388360				·3450346			
		145487	1132	49		27305	725	6
·42	·5533847				·3423041			
		146668	1181	51		28035	730	6
·43	·5680515				·3395005			
		147900	1232	54		28772	736	6
·44	·5828415				·3366233			
		149186	1286	56		29514	743	7
·45	·5977601				·3336719			
		150529	1342	59		30264	749	7
·46	·6128130				·3306455			
		151930	1402	62		31020	756	7
·47	·6280060				·3275435			
		153394	1464	65		31783	763	7
·48	·6433454				·3243652			
		154923	1529	69		32554	771	8
·49	·6588377				·3211098			
		156521	1598	72		33333	779	8
·50	·6744898		1670		·3177766		787	

TABLES III. AND IV.—(*continued*).

a	x	Δ +	Δ^2 +	Δ^3 +	z	Δ −	Δ^2 −	Δ^3 −
·50	·6744898				·3177766			
		158191	1670			34119	787	
·51	·6903088			76	·3143646			9
		159937	1747			34915	795	
·52	·7063026			81	·3108732			9
		161765	1828			35719	804	
·53	·7224791			86	·3073013			9
		163678	1913			36532	814	
·54	·7388468			91	·3036481			10
		165682	2004			37356	823	
·55	·7554150			96	·2999125			10
		167782	2100	102		38189	834	11
·56	·7721932				·2960936			
		169984	2203	109		39034	844	11
·57	·7891917				·2921902			
		172296	2312	116		39889	856	12
·58	·8064212				·2882013			
		174724	2428	124		40757	867	12
·59	·8238936				·2841256			
		177276	2552	133		41637	880	13
·60	·8416212				·2799619			
		179961	2685	143		42530	893	14
·61	·8596174				·2757089			
		182789	2828	153		43437	907	15
·62	·8778963				·2713653			
		185771	2981	165		44358	921	15
·63	·8964734				·2669295			
		188917	3147	178		45295	937	16
·64	·9153651				·2624000			
		192242	3325	193		46247	953	17
·65	·9345893				·2577753			
		195760	3518	209		47217	970	18
·66	·9541653				·2530535			
		199486	3727	227		48205	988	19
·67	·9741139				·2482330			
		203440	3954	248		49213	1007	20
·68	·9944579				·2433117			
		207641	4201	271		50240	1028	22
·69	1·0152220				·2382877			
		212114	4472	297		51289	1049	23
·70	1·0364334				·2331588			
		216882	4769	326		52362	1072	25
·71	1·0581216				·2279226			
		221977	5095	360		53459	1097	26
·72	1·0803193				·2225767			
		227432	5455	399		54582	1123	28
·73	1·1030626				·2171185			
		233286	5854	443		55734	1152	30
·74	1·1263911				·2115451			
		239583	6297	495		56916	1182	33
·75	1·1503494				·2058535			
		246374	6792	555		58130	1215	35
·76	1·1749868				·2000405			
		253721	7347	625		59380	1250	38
·77	1·2003589				·1941024			
		261693	7972	709		60669	1288	42
·78	1·2265281				·1880356			
		270373	8681	808		61999	1330	45
·79	1·2535654				·1818357			
		279861	9488	926		63374	1375	50
·8.	1·2815516		10414		·1754983		1425	

VARIATION IN "EUPAGURUS PRIDEAUXI" (HELLER).

By E. H. J. SCHUSTER, B.A.

I.

Introduction. The work on which the following paper was based was done in the winter of 1901—1902 during my occupancy of the Oxford University table at the Naples Biological Station; and I take this opportunity of expressing my appreciation of the kindness and courtesy of those members of the staff of that institution with whom I came into contact.

The subsequent calculations were done under the direction of Professor Weldon, and I herewith tender him my sincere thanks for devoting so much time and trouble to the purpose. I have also to thank Professor Pearson for his numerous and valuable suggestions.

The work itself is an attempt to determine whether members of the species *Eupagurus prideauxi* caught in shallow water, differ with regard to certain characters from those caught in comparatively deep water.

For this purpose the following three measurements were taken:

1. From the upper articulation of the propodite with the carpopodite of the right chela to the upper and outer articulation of the propodite with the dactylopodite, called Measurement No. 1. Fig. 1, *AB*.

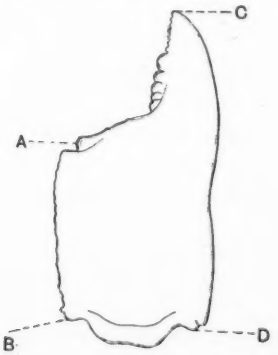


Fig. 1.

2. The length of the right chela measured from its lower articulation with the carpopodite, to the furthest point of the fixed blade of the scissors. Fig. 1, *CD*.

3. The length of the carapace along the median line.

These measurements were taken with a pair of dividers provided with a screw fine adjustment and an ivory scale divided into $\frac{1}{2}$ millimetres, they are believed to be accurate to $\frac{1}{10}$ of a millimetre.

From Measurements No. 1 and No. 2 an index called the chela index was deduced, which is intended to represent the proportion which the claw bears to the whole length of the chela.

$$\text{The chela index} = \frac{\text{Measurement No. 2} - \text{Measurement No. 1}}{\text{Measurement No. 2}}.$$

About two thousand individuals were measured, which were separated into two main groups:

I. *Shallow water forms*, from a depth of 35 metres or under;

II. *Deep water forms*, from a depth of over 35 metres.

Owing to the great difference in size between the sexes each of these groups was again subdivided into male and female. Thus finally we have to deal with four sets with about five hundred individuals in each.

II.

Comparison between deep water forms and shallow water forms as regards each measurement and the chela index taken separately. The deep water males are in each case compared with the shallow water males and the deep water females with the shallow water females.

TABLE I.

Measurement No. 1. ♂.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	9·7078 mm.	·0847 mm.	2·7588 mm.
Shallow water forms	10·2718 mm.	·0745 mm.	2·5901 mm.
Difference	0·5640 mm.	Probable error of difference = ·1128 mm.	

This measurement is greater in the shallow water forms by an amount exactly five times as great as its probable error, and therefore almost certainly significant.

TABLE II.

Measurement No. 1. ♀.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	7·4000 mm.	·0330 mm.	1·0607 mm.
Shallow water forms	7·4850 mm.	·0293 mm.	1·0234 mm.
Difference	0·0850 mm.	Probable error of difference = ·0441 mm.	

This measurement is greater in the shallow water forms by an amount about twice as great as its probable error.

TABLE III.

Right Chela Length. ♂.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	17·9676 mm.	·1451 mm.	4·7279 mm.
Shallow water forms	18·6773 mm.	·1259 mm.	4·3769 mm.
Difference	·7097 mm.	Probable error of difference = ·1922 mm.	

This measurement is greater in the shallow water forms by an amount between three and four times as great as its probable error, and therefore is probably significant.

TABLE IV.

Right Chela Length. ♀.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	14·1415 mm.	·0612 mm.	1·9679 mm.
Shallow water forms	13·9735 mm.	·0522 mm.	1·8215 mm.
Difference	·1680 mm.	Probable error of difference = ·0805 mm.	

This measurement is greater in the deep water forms by an amount twice as great as its probable error.

To sum up with regard to the absolute measurements of the right chela; in the male they are both significantly greater in the shallow water than in the deep water forms; while in the female Measurement No. 1 is greater in the shallow water forms, Measurement No. 2 in the deep water forms, and in neither case is the difference, though well marked, sufficiently great to be called significant.

TABLE V.

Chela Index. ♂.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	·461348	·000538	·017537
Shallow water forms	·451045	·000472	·016417
Difference	·010303	Probable error of difference = ·000715	

The chela index is greater in the deep water forms by an amount more than 13 times as big as its probable error, and therefore certainly significant.

TABLE VI.

Chela Index. ♀.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	·475051	·000458	·014721
Shallow water forms	·471312	·000385	·013429
Difference	·003739	Probable error of difference = ·000598	

The chela index is greater in the deep water forms by an amount more than six times as big as its probable error, and therefore almost certainly significant.

Thus Tables V. and VI. show that in the forms taken from deep water, both male and female, a greater proportion of the whole length of the chela is taken up by the blade of the scissors than in forms taken from shallow water.

TABLE VII.

Carapace Length. ♂.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	8·5854 mm.	·0512 mm.	1·6696 mm.
Shallow water forms	8·4063 mm.	·0425 mm.	1·4918 mm.
Difference	·1791 mm.	Probable error of difference = ·0665 mm.	

The mean carapace length in the deep water forms is greater than that of the shallow water forms by an amount more than twice as great as its probable error.

TABLE VIII.

Carapace Length. ♀.

	Mean	Probable error of mean	Standard Deviation
Deep water forms ...	7·5405 mm.	·0293 mm.	·9417 mm.
Shallow water forms	7·1222 mm.	·0247 mm.	·8631 mm.
Difference	·4183 mm.	Probable error of difference = ·0383 mm.	

The mean carapace length in the deep water forms is greater than that of the shallow water forms by an amount about ten times as great as its probable error.

Thus in both male and female the mean carapace length is greater in the deep water forms, in the female by a certainly significant amount, in the male by a quantity which taken alone could hardly be taken as significant, but viewed in the light of the result obtained by the female it may possibly be considered to be so.

III.

On the Comparative Variability of Deep and Shallow Water Forms.

TABLE VIII *bis.*

Table	Deep Water		Shallow Water		Difference of	
Character, Sex	S. D.	C. of V.	S. D.	C. of V.	S. D.'s	C.'s of V.
Measurement No. 1 ♂	2·7588 ± ·0599	28·418 ± ·665	2·5901 ± ·0527	25·216 ± ·544	·1687 ± ·0798	3·202 ± ·865
" " ♀	1·0607 ± ·0233	14·334 ± ·322	1·0234 ± ·0207	13·673 ± ·278	·0373 ± ·0312	·661 ± ·425
Right Chela Length ♂	4·7279 ± ·1026	26·313 ± ·609	4·3769 ± ·0890	23·434 ± ·502	·3510 ± ·1359	2·879 ± ·789
" " ♀	1·9679 ± ·0433	13·916 ± ·312	1·8215 ± ·0369	13·036 ± ·269	·1464 ± ·0569	·880 ± ·412
Carapace Length ♂ ...	1·6696 ± ·0362	19·446 ± ·438	1·4918 ± ·0303	17·746 ± ·372	·1778 ± ·0470	1·700 ± ·575
" " ♀ ...	·9417 ± ·0207	12·489 ± ·279	·8631 ± ·0175	12·118 ± ·249	·0786 ± ·0271	·371 ± ·374
Chela Index ♂	·01754 ± ·00038	38·013 ± ·936	·01642 ± ·00033	36·404 ± ·833	·00112 ± ·00051	1·608 ± 1·253
" " ♀	·01472 ± ·00032	30·988 ± ·744	·01343 ± ·00027	28·493 ± ·622	·00129 ± ·00042	2·495 ± ·970

The table above gives the standard deviations and the coefficients of variation* with their probable errors, and we see:

(i) That in both forms the male, whether we judge by standard deviation or coefficient of variation, is, for the characters considered, much more variable than the female.

(ii) That there are significant differences in the variability of the deep and shallow water forms. The deep water forms are in every single case the more variable, however variability be estimated. For males the difference is always greater and often much greater than its probable error; for females the difference is less marked, but none the less quite obvious.

We can therefore conclude that the conditions of life are probably far more stringent for the shallow than the deep sea forms, and for the females than for the males. It would, perhaps, be rash to assert that the shore crabs are a selection from the deep sea form, but the facts as to variability are not only compatible with but indeed suggestive of such an hypothesis.

IV.

Comparison of the correlation between the length of carapace and each of the other measurements for deep and shallow water forms.

To commence with the consideration of the males, Table IX. shows the correlation between the length of the carapace and Measurement No. 1 for deep, Table X. for shallow water forms.

For the former $r = \cdot 9458 \pm \cdot 0032$,
For the latter $r = \cdot 9337 \pm \cdot 0037$,
The difference $= \cdot 0120 \pm \cdot 0049$.

* Percentage variation on the mean, i.e. 100 × standard deviation and divided by the mean.

TABLE IX.

Deep Water Forms. ♂. *Correlation between Length of Carapace and Measurement No.* 1.

Measurement No. 1 (millimetres).

Length of Carapace (millimetres).	4·1– 4·5	4·6– 5·0	5·1– 5·5	5·6– 6·0	6·1– 6·5	6·6– 7·0	7·1– 7·5	7·6– 8·0	8·1– 8·5	8·6– 9·0	9·1– 9·5	9·6–10·0	10·1–10·5	10·6–11·0	11·1–11·5	11·6–12·0	12·1–12·5	12·6–13·0	13·1–13·5	13·6–14·0	14·1–14·5	14·6–15·0	15·1–15·5	15·6–16·0	16·1–16·5	16·6–17·0	17·1–17·5	Totals
12·1—12·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	4	3	1	3	1	1	2
11·6—12·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	6	5	2	—	—	—	1	15
11·1—11·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	11	6	6	4	—	—	1	—	—	18
10·6—11·0	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	1	2	3	2	—	1	—	1	—	—	—	33
10·1—10·5	—	—	—	—	—	—	—	—	—	1	—	—	1	—	1	7	7	3	3	—	—	—	—	—	—	—	—	27
9·6—10·0	—	—	—	—	—	—	—	—	—	—	—	1	5	11	18	8	4	2	3	—	—	—	—	—	—	—	—	52
9·1— 9·5	—	—	—	—	—	1	—	2	—	—	2	2	17	15	5	1	3	3	—	—	—	—	—	—	—	—	—	51
8·6— 9·0	—	—	—	—	—	—	—	—	2	2	11	17	15	1	1	—	—	—	—	—	—	—	—	—	—	—	—	49
8·1— 8·5	—	—	—	1	—	1	1	1	7	18	15	6	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
7·6— 8·0	—	—	2	—	—	—	1	6	21	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37
7·1— 7·5	—	—	—	2	—	2	14	20	10	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	49
6·6— 7·0	—	—	—	1	3	10	23	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	40
6·1— 6·5	—	—	1	3	11	7	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
5·6— 6·0	1	—	2	12	8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
5·1— 5·5	—	2	5	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
4·6— 5·0	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
Totals	2	3	11	22	22	21	40	32	40	27	31	26	39	27	25	17	15	11	17	12	12	14	5	5	4	1	2	483

TABLE X.

Shallow Water Forms. ♂. *Correlation between Length of Carapace and Measurement No.* 1.

Measurement No. 1 (millimetres).

Length of Carapace (millimetres).	4·1– 4·5	4·6– 5·0	5·1– 5·5	5·6– 6·0	6·1– 6·5	6·6– 7·0	7·1– 7·5	7·6– 8·0	8·1– 8·5	8·6– 9·0	9·1– 9·5	9·6–10·0	10·1–10·5	10·6–11·0	11·1–11·5	11·6–12·0	12·1–12·5	12·6–13·0	13·1–13·5	13·6–14·0	14·1–14·5	14·6–15·0	15·1–15·5	15·6–16·0	16·1–16·5	16·6–17·0	17·1–17·5	Totals
11·6—12·0	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	3
11·1—11·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	3	1	—	9
10·6—11·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	3	3	3	8	3	2	2	1	—	—	28
10·1—10·5	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	2	2	8	13	7	4	5	—	—	—	—	43
9·6—10·0	—	—	—	—	—	—	—	—	—	1	—	—	—	1	3	7	11	6	9	13	1	—	—	—	—	—	—	52
9·1— 9·5	—	—	—	—	—	—	—	—	—	—	1	2	3	6	14	12	13	5	2	—	—	—	—	—	—	—	—	58
8·6— 9·0	—	—	—	—	—	—	1	—	1	—	2	5	11	16	15	7	4	1	—	—	—	—	—	—	—	—	—	63
8·1— 8·5	—	—	—	—	1	1	—	—	2	4	18	17	24	9	—	—	—	1	—	—	—	—	—	—	—	—	—	77
7·6— 8·0	—	—	—	—	—	—	1	—	7	16	22	10	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	57
7·1— 7·5	—	—	—	—	—	—	4	12	24	12	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	59
6·6— 7·0	—	—	—	—	1	3	9	13	11	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39
6·1— 6·5	—	—	—	—	6	8	9	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
5·6— 6·0	1	—	—	8	7	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	20
5·1— 5·5	—	—	—	10	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
4·6— 5·0	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
4·1— 4·5	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
Totals	2	4	1	18	17	16	24	26	45	37	50	34	38	33	33	27	32	18	22	29	16	7	8	6	5	2	—	550

TABLE XI.

Deep Water Forms. ♂. *Correlation between Length of Right Chela and Length of Carapace.*

Length of Right Chela (millimetres).

Length of Carapace (millimetres).	8·0– 8·9	9·0– 9·9	10·0–10·9	11·0–11·9	12·0–12·9	13·0–13·9	14·0–14·9	15·0–15·9	16·0–16·9	17·0–17·9	18·0–18·9	19·0–19·9	20·0–20·9	21·0–21·9	22·0–22·9	23·0–23·9	24·0–24·9	25·0–25·9	26·0–26·9	27·0–27·9	28·0–28·9	29·0–29·9	30·0–30·9	Totals
12·1—12·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2
11·6—12·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	3	—	—	—	—	15
11·1—11·5	—	—	—	—	—	—	—	—	—	1	—	—	—	2	1	—	3	5	6	2	—	—	1	18
10·6—11·0	—	—	—	—	—	—	—	—	—	—	—	1	—	6	6	1	15	4	4	3	—	—	—	33
10·1—10·5	—	—	—	—	—	—	—	—	—	—	—	5	18	16	9	3	3	3	1	—	—	—	—	27
9·6—10·0	—	—	—	—	—	1	2	—	—	2	6	20	11	3	6	—	1	—	—	—	—	—	—	52
9·1— 9·5	—	—	—	—	—	—	—	1	1	16	24	5	1	1	—	—	—	—	—	—	—	—	—	51
8·6— 9·0	—	—	—	—	—	2	5	4	20	20	3	1	—	—	—	—	—	—	—	—	—	—	—	49
8·1— 8·5	—	—	1	—	1	1	23	17	12	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
7·6— 8·0	—	1	—	2	—	8	7	14	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37
7·1— 7·5	—	—	—	—	8	25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	49
6·6— 7·0	—	—	—	—	9	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	40
6·1— 6·5	—	1	1	9	9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
5·6— 6·0	1	—	7	13	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
5·1— 5·5	—	4	6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
4·6— 5·0	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
Totals	2	8	15	25	19	40	37	37	35	39	34	32	31	28	22	11	22	14	14	11	5	—	2	483

TABLE XII.

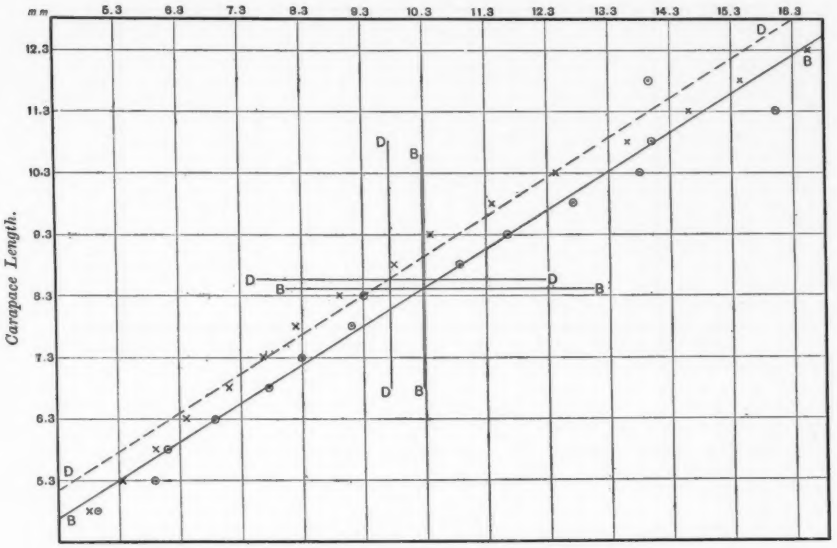
Shallow Water Forms. ♂. *Correlation between Length of Right Chela and Length of Carapace.*

Length of Right Chela (millimetres).

Length of Carapace (millimetres).	6·8 –0·8	6·9 –0·9	10·0–10·9	11·0–11·9	12·0–12·9	13·0–13·9	14·0–14·9	15·0–15·9	16·0–16·9	17·0–17·9	18·0–18·9	19·0–19·9	20·0–20·9	21·0–21·9	22·0–22·9	23·0–23·9	24·0–24·9	25·0–25·9	26·0–26·9	27·0–27·9	28·0–28·9	29·0–29·9	Totals
11·6—12·0	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	—	1	—	3
11·1—11·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	4	3	—	—	9
10·6—11·0	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	2	4	7	10	1	1	—	28
10·1—10·5	—	—	—	—	—	—	—	—	1	—	—	—	—	1	3	7	12	11	7	1	—	—	43
9·6—10·0	—	—	—	—	—	—	—	—	—	1	—	—	2	8	19	7	13	2	—	—	—	—	52
9·1— 9·5	—	—	—	—	—	—	—	—	—	2	2	3	15	20	13	3	—	—	—	—	—	—	58
8·6— 9·0	—	—	—	—	—	—	1	—	2	2	11	21	17	6	3	—	—	—	—	—	—	—	63
8·1— 8·5	—	—	—	—	2	—	—	1	7	21	34	11	—	1	—	—	—	—	—	—	—	—	77
7·6— 8·0	—	—	—	—	—	1	—	7	27	17	4	1	—	—	—	—	—	—	—	—	—	—	57
7·1— 7·5	—	—	—	—	—	1	14	30	11	3	—	—	—	—	—	—	—	—	—	—	—	—	59
6·6— 7·0	—	—	—	1	2	9	20	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39
6·1— 6·5	—	—	—	1	11	12	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
5·6— 6·0	1	—	2	15	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	20
5·1— 5·5	—	—	5	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
4·6— 5·0	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
4·1— 4·5	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
Totals	3	3	8	24	17	23	35	45	48	47	51	36	35	36	40	19	29	20	19	7	4	1	550

Diagram I. shows the regression lines of Measurement No. 1 on the carapace for these two depths; the dotted line is for the deep water forms, the continuous line for the shallow water forms. It will be noticed that one carapace length is associated with a much smaller value of Measurement No. 1 in the deep water forms than in the shallow.

DIAGRAM I. *Measurement No. 1.*



Comparison between Regression Lines of Measurement No. 1 on Carapace Length between Shallow Water ♂♂ (BB) and Deep Water ♂♂ (DD).

Shallow Water ♂♂.
Mean Measurement No. 1, 10·2718 mm.
Mean Carapace Length 8·4063 mm.
Regression of Measurement No. 1 on Carapace Length 1·6212.

Deep Water ♂♂.
Mean Measurement No. 1, 9·7078.
Mean Carapace Length 8·5854.
Regression of Measurement No. 1 on Carapace Length 1·5628.

Table XI. shows the correlation between the length of the right chela and the length of the carapace for deep, Table XII. for the shallow water ♂ forms.

For the deep water forms $r = \cdot 9389 \pm \cdot 0036$,
For the shallow water forms $r = \cdot 9503 \pm \cdot 0028$,
Difference $= \cdot 0114 \pm \cdot 0046$.

TABLE XIII.

Deep Water Forms. ♂. *Correlation between Carapace Length and Chela Index.*

Chela Index.

Carapace Length (millimetres).	386–395	396–405	406–415	416–425	426–435	436–445	446–455	456–465	466–475	476–485	486–495	496–505	506–515	Totals
12·1—12·5	—	—	—	1	1	—	—	—	—	—	—	—	—	2
11·6—12·0	—	1	2	1	3	4	3	1	—	—	—	—	—	15
11·1—11·5	—	—	2	1	1	6	6	1	1	—	—	—	—	18
10·6—11·0	1	1	—	1	5	8	10	3	4	—	—	—	—	33
10·1—10·5	—	—	1	1	3	6	7	6	1	2	—	—	—	27
9·6—10·0	—	—	—	—	2	7	11	13	11	5	2	1	—	52
9·1— 9·5	—	—	—	3	—	6	10	16	11	4	1	—	—	51
8·6— 9·0	—	—	—	—	—	6	13	11	10	5	4	—	—	49
8·1— 8·5	—	—	—	—	—	—	11	9	15	9	7	—	—	51
7·6— 8·0	—	—	—	—	—	1	5	8	13	7	2	—	1	37
7·1— 7·5	—	—	—	—	—	1	3	16	19	7	3	—	—	49
6·6— 7·0	—	—	—	—	—	1	8	6	12	7	5	1	—	40
6·1— 6·5	—	—	—	—	—	—	—	4	11	8	—	—	—	23
5·6— 6·0	—	—	—	—	—	1	1	2	8	7	3	—	1	23
5·1— 5·5	—	—	—	—	1	1	1	2	4	1	—	—	—	10
4·6— 5·0	—	—	—	—	—	—	—	1	1	1	—	—	—	3
Totals	1	2	5	8	15	48	89	99	122	63	27	2	2	483

TABLE XIV.

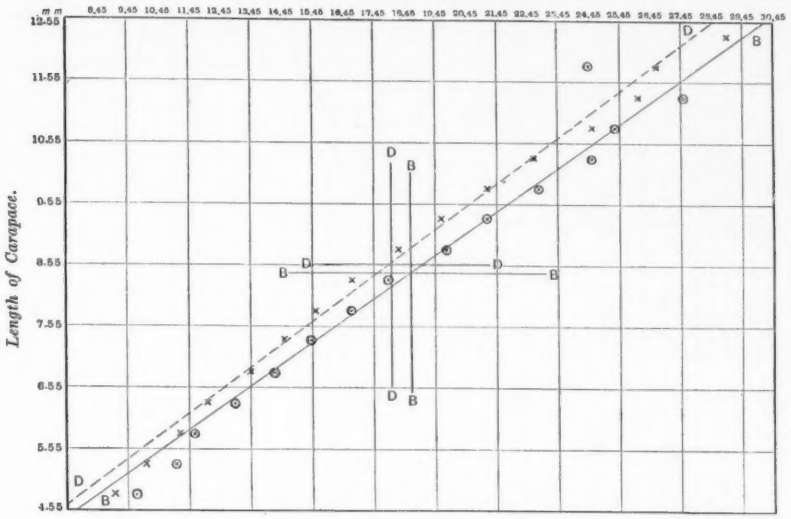
Shallow Water Forms. ♂. *Correlation between Length of Carapace and Chela Index.*

Chela Index.

Carapace Length (millimetres).	396–405	406–415	416–425	426–435	436–445	446–455	456–465	466–475	476–485	486–495	496–505	Totals
11·6—12·0	—	—	1	1	—	—	—	—	1	—	—	3
11·1—11·5	1	2	4	2	—	—	—	—	—	—	—	9
10·6—11·0	—	2	4	3	9	4	5	1	—	—	—	28
10·1—10·5	1	2	4	6	14	10	6	—	—	—	—	43
9·6—10·0	—	2	5	9	14	11	6	4	—	1	—	52
9·1— 9·5	—	1	1	7	14	16	9	8	2	—	—	58
8·6— 9·0	1	—	4	14	8	17	12	2	3	2	—	63
8·1— 8·5	—	1	—	4	18	22	19	7	4	2	—	77
7·6— 8·0	—	—	—	4	5	17	17	11	3	—	—	57
7·1— 7·5	—	—	1	—	6	13	26	8	3	1	1	59
6·6— 7·0	—	—	—	1	7	13	12	5	1	—	—	39
6·1— 6·5	—	—	—	—	2	5	8	6	2	1	—	24
5·6— 6·0	—	—	—	1	2	6	5	3	3	—	—	20
5·1— 5·5	—	—	—	1	1	1	2	4	2	1	—	12
4·6— 5·0	—	—	—	—	—	—	1	1	1	—	—	3
4·1— 4·5	—	—	—	—	—	1	1	1	—	—	—	3
Totals	3	10	24	53	100	136	129	61	25	8	1	550

Diagram II. shows the regression lines of the chela length on the carapace for deep and shallow water forms, the dotted line representing that of the deep water forms as in Diagram I. The diagram shows that the same carapace length is associated with a smaller chela length in the deep water forms than in the shallow, but the difference between deep and shallow is not so marked as in Diagram I.

DIAGRAM II. *Length of Right Chela.*



Comparison between Regression Lines of Right Chela Length on Carapace Length between Shallow Water ♂♂ (BB) and Deep Water ♂♂ (DD).

Shallow Water ♂♂.
Mean Length of Right Chela 18·6773 mm.
Mean Length of Carapace 8·4063 mm.
Regression of Chela on Carapace 1·405.

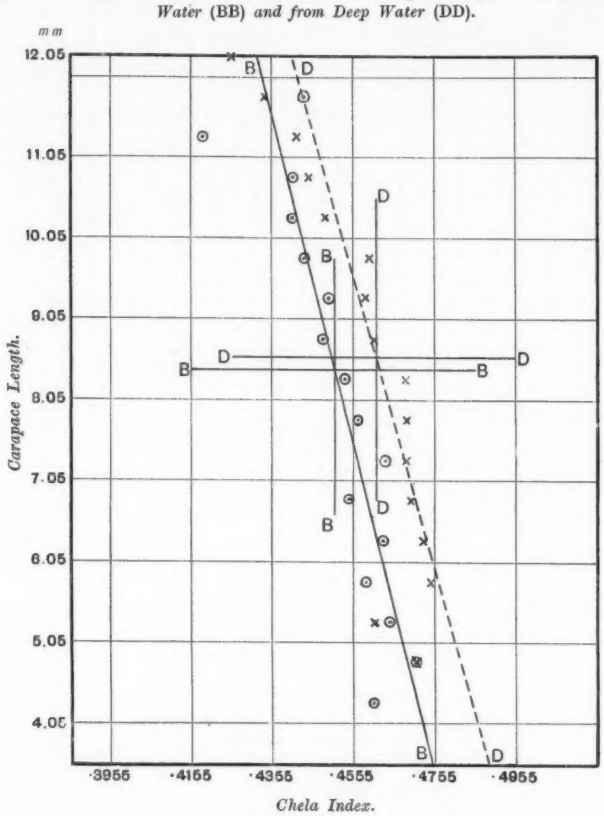
Deep Water ♂♂.
Mean Length of Right Chela 17·9676.
Mean Length of Carapace 8·5854.
Regression of Chela on Carapace 1·329.

Tables XIII. and XIV. show the correlation between carapace length and chela index for deep and shallow water groups respectively.

Deep water $r = -\cdot5224 \pm \cdot0223$,
Shallow water $r = -\cdot4448 \pm \cdot0231$,
Difference $\cdot0776 \pm \cdot0321$.

Diagram III. shows the regression lines of the chela index on the carapace length. The same carapace length is associated with a smaller chela index for the shallow water than for the deep water forms.

DIAGRAM III. *Regression Lines of Chela Index on Carapace Length in ♂♂ from Shallow Water* (BB) *and from Deep Water* (DD).



With regard to the females, Tables XV. and XVI. correspond to Tables IX. and X. of the males and show the correlation between the carapace length and Measurement No. 1 for the deep and shallow water forms respectively.

Deep water $r = \cdot 8466 \pm \cdot 0088$,
Shallow water $r = \cdot 8428 \pm \cdot 0083$,
Difference $\cdot 0037 \pm \cdot 0121$.

The difference being less than its probable error is quite negligible.

TABLE XV.

Deep Water Forms. ♀. *Correlation between Length of Carapace and Measurement No.* 1.

Measurement No. 1 (millimetres).

Length of Carapace (millimetres).	4·1– 4·5	4·6– 5·0	5·1– 5·5	5·6– 6·0	6·1– 6·5	6·6– 7·0	7·1– 7·5	7·6– 8·0	8·1– 8·5	8·6– 9·0	9·1– 9·5	9·6–10·0	10·1–10·5	Totals
9·6—10·0	—	—	—	—	—	—	—	—	—	—	1	1	—	2
9·1— 9·5	—	—	—	—	—	—	—	2	2	3	6	—	—	13
8·6— 9·0	—	—	—	1	—	—	—	2	6	18	7	—	2	36
8·1— 8·5	—	—	1	—	2	—	3	29	51	9	2	1	—	98
7·6— 8·0	—	—	3	2	1	4	35	47	9	5	1	—	—	107
7·1— 7·5	—	1	2	—	2	21	47	18	2	—	—	—	—	93
6·6— 7·0	—	1	—	—	13	30	10	2	—	—	—	—	—	56
6·1— 6·5	—	—	—	10	19	—	—	—	—	—	—	—	—	29
5·6— 6·0	1	1	7	9	4	—	—	—	—	—	—	—	—	22
5·1— 5·5	—	1	3	1	—	—	—	—	—	—	—	—	—	5
4·6— 5·0	—	6	1	—	—	—	—	—	—	—	—	—	—	7
4·1— 4·5	1	1	—	—	—	—	—	—	—	—	—	—	—	2
Totals	2	11	17	23	41	55	95	100	70	35	17	2	2	470

TABLE XVI.

Shallow Water Forms. ♀. *Correlation between Length of Carapace and Measurement No.* 1.

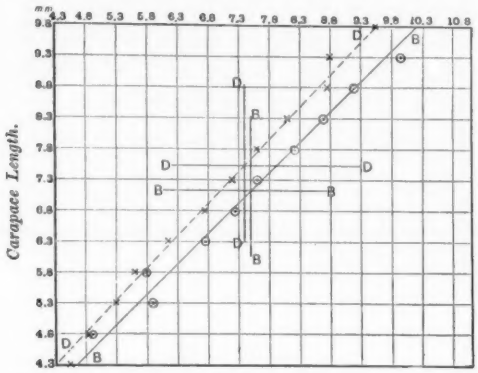
Measurement No. 1 (millimetres).

Length of Carapace (millimetres).	4·1– 4·5	4·6– 5·0	5·1– 5·5	5·6– 6·0	6·1– 6·5	6·6– 7·0	7·1– 7·5	7·6– 8·0	8·1– 8·5	8·6– 9·0	9·1– 9·5	9·6–10·0	10·1–10·5	10·6–11·0	11·1–11·5	Totals
9·1—9·5	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	3
8·6—9·0	—	—	—	—	—	—	—	—	1	5	8	2	—	—	—	16
8·1—8·5	—	—	—	—	1	—	1	3	12	30	9	1	—	—	—	57
7·6—8·0	—	—	—	1	1	2	8	25	46	22	2	1	—	—	—	108
7·1—7·5	—	2	—	—	—	9	34	54	19	1	—	—	—	—	—	119
6·6—7·0	—	3	—	—	4	25	62	19	2	—	—	—	—	—	—	115
6·1—6·5	—	—	1	1	19	38	14	—	—	—	—	—	—	—	—	73
5·6—6·0	2	2	3	7	25	3	—	—	—	—	—	—	—	—	—	42
5·1—5·5	—	—	—	11	2	—	—	—	—	—	—	—	—	—	—	13
4·6—5·0	1	4	3	—	—	—	—	—	—	—	—	—	—	—	—	8
Totals	3	11	7	20	52	77	119	101	80	58	20	5	—	1	—	554

Diagram IV. corresponds to Diagram I. It represents the regression lines of Measurement No. 1 on carapace length, the dotted line being that of the deep water females, the continuous line that of the shallow water females.

It will be seen that the relation these two lines bear to one another is very much the same as that between the corresponding lines of the males. The diagram shows therefore that in the females as well as the males the same carapace length is associated with a smaller value of Measurement No. 1 in the deep water forms than in the shallow water forms.

Diagram IV. *Measurement No.* 1.



Comparison between Regression Lines of Measurement No. 1 on Carapace between Shallow Water ♀♀ (BB) and Deep Water ♀♀ (DD).

Shallow Water ♀♀.
Mean Carapace Length 7·1222 mm.
Mean Measurement No. 1 7·4850 mm.
Regression ·999. $r = ·8428$.

Deep Water ♀♀.
Mean Carapace Length 7·5405.
Mean Measurement No. 1 7·4000.
Regression ·9535. $r = ·8466$.

Tables XVII. and XVIII. show the correlation between the chela length and the carapace length in the deep and shallow water females, corresponding to Tables XI. and XII. of the males.

Deep water $r = ·8626 \pm ·0080$,
Shallow water $r = ·8841 \pm ·0063$,
Difference $= ·0215 \pm ·0106$.

TABLE XVII.

Deep Water Forms. ♀. *Correlation between Length of Right Chela and Length of Carapace.*

Length of Right Chela (millimetres).

Length of Carapace (millimetres).	8·0– 8·9	9·0– 9·9	10·0–10·9	11·0–11·9	12·0–12·9	13·0–13·9	14·0–14·9	15·0–15·9	16·0–16·9	17·0–17·9	18·0–18·9	Totals
9·6—10·0	—	—	—	—	—	—	—	—	—	—	2	2
9·1— 9·5	—	—	—	—	—	—	—	3	4	4	2	13
8·6— 9·0	—	—	—	1	—	—	1	6	15	11	2	36
8·1— 8·5	—	—	1	—	2	1	15	51	25	3	—	98
7·6— 8·0	—	—	4	1	3	9	50	34	4	2	—	107
7·1— 7·5	—	1	2	—	2	43	43	2	—	—	—	93
6·6— 7·0	—	1	—	1	23	28	3	—	—	—	—	56
6·1— 6·5	—	—	1	20	8	—	—	—	—	—	—	29
5·6— 6·0	1	2	10	9	—	—	—	—	—	—	—	22
5·1— 5·5	—	2	3	—	—	—	—	—	—	—	—	5
4·6— 5·0	1	6	—	—	—	—	—	—	—	—	—	7
4·1— 4·5	2	—	—	—	—	—	—	—	—	—	—	2
Totals	4	12	21	32	38	81	112	96	48	20	6	470

TABLE XVIII.

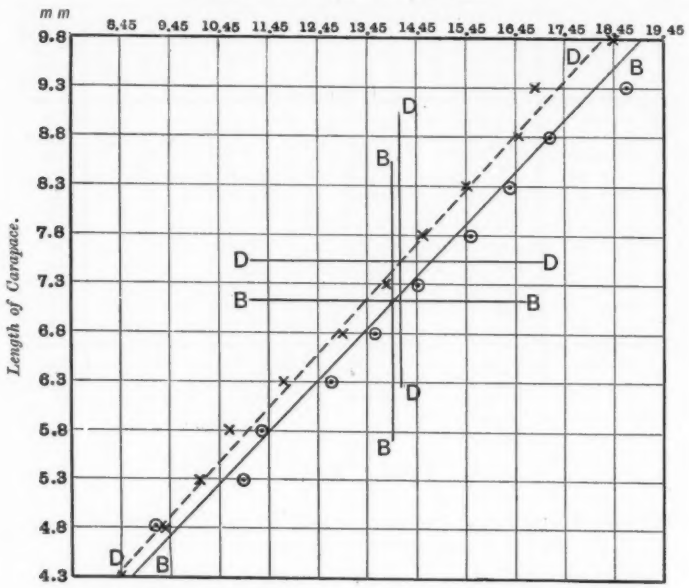
Shallow Water Forms. ♀. *Correlation between Length of Right Chela and Length of Carapace.*

Length of Right Chela (millimetres).

Length of Carapace (millimetres).	7·0– 7·9	8·0– 8·9	9·0– 9·9	10·0–10·9	11·0–11·9	12·0–12·9	13·0–13·9	14·0–14·9	15·0–15·9	16·0–16·9	17·0–17·9	18·0–18·9	19·0–19·9	Totals
9·1—9·5	—	—	—	—	—	—	—	—	—	—	—	2	1	3
8·6—9·0	—	—	—	—	—	—	—	—	1	4	9	2	—	16
8·1—8·5	—	—	—	—	—	1	—	2	11	32	10	1	—	57
7·6—8·0	—	—	—	—	1	1	2	21	56	25	2	—	—	108
7·1—7·5	—	—	2	—	—	1	23	63	29	1	—	—	—	119
6·6—7·0	—	—	1	2	—	7	71	31	2	1	—	—	—	115
6·1—6·5	—	—	—	—	8	40	24	1	—	—	—	—	—	73
5·6—6·0	—	2	2	5	22	10	1	—	—	—	—	—	—	42
5·1—5·5	—	—	—	7	6	—	—	—	—	—	—	—	—	13
4·6—5·0	—	3	4	1	—	—	—	—	—	—	—	—	—	8
Totals	—	5	9	15	37	60	121	118	99	63	21	5	1	554

Diagram V. shows the regression lines of the right chela length on the carapace length for deep and shallow water females; it corresponds to Diagram II. The two lines bear much the same relation to one another as in that diagram. And in the same way as in Diagram II. the regression lines run closer together than in Diagram I. thus those of Diagram V. run closer together than those in Diagram IV.,

DIAGRAM V. *Length of Right Chela.*



Comparison between Regression Lines of Right Chela Length on Carapace Length of Shallow Water ♀♀ (BB) and Deep Water ♀♀ (DD).

Shallow Water ♀♀.
Mean Carapace Length 7·1222 mm.
Mean Chela Length 13·9735 mm.
Regression ·9329. $r = ·8841$.

Deep Water ♀♀.
Mean Carapace Length 7·5405 mm.
Mean Chela Length 14·1415 mm.
Regression ·9013. $r = ·8626$.

showing that these points in relation to the chela and carapace for the deep and shallow water forms which are illustrated by these diagrams are much the same in the males as they are in the females. Namely that the same carapace length is associated with a smaller value for Measurement No. 1 in the deep water forms than in the shallow water forms, also with a smaller value of Measurement No. 2,

but that neither in the male nor in the female is the difference of the values of Measurement No. 2 so great as of Measurement No. 1.

It was found that in the females the value of r for the correlation between chela index and carapace length was less than its probable error and the regression practically nothing and so neither the correlation tables nor the diagram of the regression lines are given here.

V.

Comparison of Regression and Correlation Coefficients.

TABLE XIX.

Regression Coefficients of First Order.

Regression Coefficients	Male		Female	
	Deep Sea	Shallow	Deep Sea	Shallow
Measurement No. 1 on Carapace Length	1·5628	1·6212	·9535	·9994
Carapace Length on Measurement No. 1	·5723	·5378	·7516	·7108
Right Chela Length on Carapace Length	1·3293	1·4056	·9013	·9329
Carapace Length on Right Chela Length	·6633	·6477	·8255	·8379
Chela Index on Carapace Length	−·2744	−·2447	—	—
Carapace Length on Chela Index	−·9947	−·8084	—	—

Now at first sight it might appear that the greatest differences here between deep sea and shallow forms occur when we consider the regressions of the other organs on carapace length. But if we are dealing only with the question of significant differences, we must note that the probable errors are much larger in these cases. Thus the regression of right chela length on carapace length for deep sea males must be read $1{\cdot}3293 \pm {\cdot}0298$, while for the inverse regression of carapace length on right chela length we have $·6633 \pm ·0037$. Hence the change in the latter regression from shallow to deep sea forms although so much smaller is really more significant. If, however, we suppose the changes between the regression coefficients for the two forms to be due, not to the random character of the samples, but to the direct action of natural selection, the greatest absolute differences are clearly in the regressions of the other organs on the carapace length, or the suggestion naturally follows that the differentiation between the deep sea and shallow forms is due more to a direct selection of the chela than of the carapace length. It would be interesting to determine whether this suggestion is or is not strengthened when the regression coefficients of higher orders are investigated.

Exhibiting all the correlation coefficients in a single table we have

TABLE XX.

Correlation Coefficients.

Characters	Male		Female	
	Deep Sea	Shallow	Deep Sea	Shallow
Measurement No. 1 and Carapace Length	·9458	·9338	·8466	·8428
Right Chela Length and Carapace Length	·9389	·9503	·8626	·8841
Chela Index and Carapace Length ...	−·5224	−·4448	Insensible	Insensible

Or, we conclude that the male has the parts investigated far more highly correlated than the female. We have thus the males more variable and more highly correlated than the females. It is remarkable that in man the female is on the whole more variable, and more highly correlated than the male. There would appear therefore to be a relation between high variability and high correlation. Or, perhaps, it would be better to say that the more stringent the selection the smaller is the variability, and as a rule the correlation. Interpreted in this manner we might hazard the suggestion that the female crabs of both forms are subjected to more stringent selection than the males.

VI.

General conclusions. (*a*) We have seen that with respect to the three characters measured certain differences do exist between those animals which have been living in deep water and those which have been living in shallow water. These differences may be congenital or they may be produced afresh in any generation. The latter alternative is more likely than the former as the larvae are pelagic and those produced by deep water parents must get mixed up pretty effectually with those that spring from shallow water. It is of course possible that the former find their way to the deep water in settling down to the bottom and the latter to the shallow water, but this is rather difficult to imagine; or that they sink down without regard to the depth of the water, but die off immediately if they are not in approximately the same depth in which their parents lived. It is a problem of the same kind as is connected with the existence of local races in many animals with pelagic larvae.

In the case of the difference arising separately in each generation, it must be produced either by selection or by the direct action of the environment and we have no data for determining which of these is the case.

(*b*) The male is certainly more variable and more highly correlated than the female in both deep sea and shallow forms.

(*c*) In both sexes the deep sea forms are more variable than the shallow water forms.

These facts are consonant with a greater selection of female than male and of shallow water than deep sea forms.

THE LAW OF ANCESTRAL HEREDITY*.

By KARL PEARSON, F.R.S.

(1) To any one who has made a close study of heredity, whether it be in men, mice or plants, there are one or two conclusions which must be accepted at once. The first of these conclusions is:

(*a*) *That a knowledge of the characters of the parents does not accurately define the character in the offspring.*

Every one is familiar with the fact that given two pairs of parents possessing the like character in the same degree the offspring of one pair will differ from those of the other in respect of this character, and further that all the offspring of the same pair are not alike in the character; and this variation in the offspring of the same pair or of like pairs of parents may be very great indeed.

This point although familiar is frequently neglected. It is so important that it seems worth while to illustrate it by material at my disposal. The pedigrees I give are only samples of many hundreds in my possession, but they are conclusive. I turn first to coat-colour in thoroughbred horses.

(i) The mare *Ally* (bay) covered with *Little John* (bay) gave a roan colt. There are no black coats in the ancestry of either *Ally* or *Little John* back to their great-grandsires and great-grandams. *Jest Ally's* dam has, however, the *Delpini* grey colour, which accounts for the roan. *Ally* again covered, this time by *Interpreter* (bay), gives a *black* filly. There is no black blood in *Ally*, but *Interpreter's* paternal grandsire was the well-known black horse, *Socerer*. Thus two bay horses may give a roan, or two bays may give a black foal. I can cite any number of instances, of course, in which they give bay, brown or chestnut offspring.

* The following paper is put together not as a reply to Mr W. Bateson's rhetorical attack on the Law of Ancestral Heredity (published in his recent book *Mendel's Principles of Heredity*), but simply to indicate to those interested in the matter what are really the fundamental assumptions involved in the Law of Ancestral Heredity, and how far it enables us to describe actually observed experience in man, horse and dog, which I am unable under any hypothesis to bring under Mendel's "Principles." In the course of this year various investigations on heredity in both plant and animal life will I hope be published, and these will tend to throw further light on the laws of heredity.

(ii) *Socerer* was a black horse, son of the black *Trumpator*, out of *Young Giantess*, a bay. But the four grandparents of *Socerer* were two chestnuts, a brown and a bay, and of the eight great-grandparents seven were bays and of one, the *Snap Mare*, I have not been able to ascertain the colour. Where did *Socerer* and *Trumpator* get their blackness from? *Snap* was brown, but his maternal grandsire was the black horse *Gypsy*. In the same way *Rinaldo* black was born of the chestnut *Whiskey Mare* covered by the brown stallion *Milo*, but in the fourth and fifth generation backwards comes in the black *Gypsy* blood through the brown stallion *Snap*.

(iii) A bay filly is the result of crossing the chestnut stallion *General Graham* and the grey *Beningborough Mare*. The chestnut stallion *Prophet* and the grey mare *Virago* give a grey filly. The chestnut stallion *Woodpecker* and the grey *Herod Mare* give a chestnut filly *Chestnut Skin*. Clearly the coat-colour of the parents will not define that of the offspring. The bay filly, however, of the *General Graham* and *Beningborough Mare* cross is elucidated when we know that while it had one chestnut and one grey grandparent, it had two bays for grandparents.

I could multiply these pedigrees indefinitely, but the above will be sufficient to demonstrate the point for horses, i.e. the coat-colour of a horse may be unintelligible unless we examine the ancestry. We may easily find offspring of all shades from grey to black, whose parents had the same colour, say both bays.

I now pass to dogs. Let us first take Basset Hounds. Here the colours are lemon, white and black, and if they all occur the hound is termed tricolour.

(i) In 1885 the tricolour bitch *La Fanfare* put to the tricolour dog *Bourbon* gave the tricolour pup *Bluette*,—melanism appeared in the offspring. In the same year the tricolour bitch *Queen Dido* covered by the tricolour dog *Bourbon* gave the lemon and white pup *Blonde*, or melanism disappeared. In 1887 the lemon and white bitch *Jessie* crossed with the lemon and white dog *K. Bendigo* gave the tricolour hound *Bendigo II*. Thus when melanism is present in the parents it may be absent in the children, or when absent in the parents, it may reappear in the children.

(ii) *His Lordship* was tricolour, but his parents *Scipio* and *Fama* were only lemon and white. His black becomes quite explicable when we note that of his four grandparents two were tricolour.

(iii) Or, in the same litter, the tricolour bitch *Iris* by the tricolour dog *Count* casts five hounds of which three are tricolour and two lemon and white only. It is thus impossible to predict from the character of the parents whether black will or will not appear in the offspring. But here again ancestry throws light on the matter. *Count's* parents *Ugly* and *Rosalind II.* were both lemon and white and *Count* got his black from his paternal grandparents, who were both tricolour. Of *Iris'* parents one showed melanism and the other did not.

What we are dealing with here is not a slight variation in shade, but the presence or total absence of black pigmentation. This point must be borne in mind.

These cases might be enough, perhaps, for dogs, but I cite the following instances out of many for greyhounds *.

I would first state that I have sought in vain for any single colour or combination of colours which breeds true to itself. There is as far as I have been able to ascertain absolutely nothing that can correspond to Mendel's "recessive" character. At first it seemed to me possible that black would be such. *Bessie Hay*, a black bitch covered by *Grinlaw* a black dog, had a litter of entirely black offspring in 1896, and repeated this performance with four instead of five whelps in 1897. But the black bitch *Finest Fury* covered by the black dog *Petronius* in a litter of eight had five black and three red offspring. Black *Middleton* and black *Raven* gave three black, one blue, a red and a fawn offspring, while the black bitch *Mayfly* crossed by the black *Hermas* produced four black and two fawn offspring. In fact almost every possible greyhound colour will be found in the offspring of two black parents,—black, blue, either of these and white, fawn, red, brindled! If we next take such a cross as black and red we find red *Recoil* covered by black *Fortuna Favente* giving six black offspring and we at once consider whether black may be "dominant." But the red bitch *Maid of all Work* threw in a litter to the black dog *Black Marauder* six brindled, one black, one blue offspring and a white bitch with one eye and two ears black. Black *Nebula* crossed by red *Herschel* gave five fawn offspring, one black and one blue and white. Parents of the same colours will be found to give litters of red, and black and white offspring. A blue sire (*Blue Peter VIII.*) and a black dam (*Flotilla*) will produce all the offspring red, while a black bitch covered by a black and white sire will give pure black dogs, pure white dogs, or mixtures of black, red or fawn with white. The black bitch *Queen of the Colonies* covered by the red fawn dog *Dodger* gave beside five black and three red offspring a white bitch. No single colour breeds true to itself, and while all these results are explicable on the basis of ancestry, the Mendelian can only pass them over by asserting that the greyhound is an incorrigible mongrel.

We may next pass to eye-colour in man and I give the following instances†. Father and mother both blue, two out of six children dark brown. Father and mother both blue, one out of six children dark brown (colour of father's maternal grandfather). Father and mother both light grey, four children light grey, and five black or very dark brown (colour of maternal grandmother). Now turn to the opposite end of the scale: Father and mother black or very dark brown,

* Pedigrees taken from data collected by Mr Howard Collins.

† I gave the substance of such pedigrees to Prof. Weldon, who made use of it in his article on Mendel's Law in *Biometrika* (Vol. I. p. 242) in a passage criticised by Mr Bateson in his *Defence of Mendel*, p. 192. I first tested Mendel's theory of dominance on eye-colour in man when I read Mendel's paper in 1900 and then communicated the result to Professor Weldon.

one of eight children light blue (colour of paternal grandfather). Father black and mother dark brown, one child out of three hazel (the colour of paternal grandmother). Father and mother very dark brown or black, two children blue and one black. Father and mother very dark brown or black, two out of four children blue (colour of maternal grandmother). It is clear that neither end of the eye-colour scale gives a recessive character which breeds true to itself. Now note what we get if we attempt to select for several generations *paying attention to ancestry*. Father and mother blue, all four grandparents blue, and the five recorded great-grandparents blue, all four children blue. Parents and grandparents all blue, all eight children blue. Father and mother light grey, and the four grandparents all light grey or blue, all seven children grey. Father and mother grey and all the known ancestry of both grey or blue, all four children grey or blue. Father and mother both dark brown, all known ancestry brown for both, out of nine children seven dark brown, one brown and one light brown. Father and mother hazel, all ancestry to great-grandparents hazel with the exception of a blue-eyed grandfather, all five children hazel.

Lastly let us look at crosses between blue and dark eyes. Father light blue and mother dark brown, three children blue, four children dark brown. Father dark brown and mother blue, two children blue and four brown-eyed. Father dark brown and mother blue, three brown and six blue-eyed children; here all mother's known ancestry was blue. If blue were "dominant" such results are impossible unless we suppose every dark brown parent above recorded was a hybrid. If blue be "recessive" it ought to breed true, but we have seen that it does not. But as a matter of fact we can pick out cases from the record of one blue and one very dark parent giving almost every conceivable result, e.g. father light blue and mother black, four children blue and one light grey. Father brown and mother blue, eight children ranging from brown to dark brown. In all cases an examination of ancestry throws light, if it does not fully explain in each case what has taken place. It is clear from these cases that parentage will not enable us to predict offspring definitely. But what is also equally clear is that if all the ancestry be the same the offspring may differ among themselves considerably, e.g. by the presence or total absence of black pigmentation. Hence we reach our second conclusion:

(*b*) *That a knowledge of the whole ancestry while it certainly limits the range of variation does not absolutely define the character of the offspring.*

(2) Before we leave the cases above it is just worth reiterating that nothing corresponding to Mendel's principles appears in these characters for horses, dogs, or men. If black or grey coat-colour in horses were "recessive," when two blacks were mated we should expect only black offspring, but black can disappear for a generation or even two and then reappear. Or, take a case like that of a grey horse *Viscount*, where grey remained dominant for three generations only to disappear before the chestnut of the mare *Blue Stocking* in the *Viscount* and *Blue-Stocking* filly *Miss Johanna*! It is the same with every coat-colour taken, its

relative constancy depends largely on the extent to which it has appeared in the ancestry, and one by one black, bay, chestnut, grey must be dismissed by the Mendelian as neither "recessive" nor "dominant," but as marking "permanent and incorrigible mongrels." The same remark applies to coat-colour in hounds, black must be the mark of incorrigible mongrelism for it is neither recessive nor dominant. It is the same with red and white, no colour in the parents gives for every case consistent offspring.

Nor again do we find that two black-eyed human beings nor two blue-eyed human beings mated together give any sign of the dominance of one or other of these extremes; any more than do intermediate tints to which the description of "incorrigible mongrelism" may by some be applied. It will be clear that for man (and this applies not only to eye-colour, but to hair-colour, and to a whole series of measurable characters * of which we have in each case more than a thousand instances in our Family Records), for horses in coat-colour, and for dogs in coat-colour † nothing approaching Mendelian principles holds. We have therefore to classify large portions of the animal kingdom as exceptions to Mendel's Laws,—and these are cases where the evidence is not based on five or ten individual crosses followed perhaps for two or at most three generations—but on 1000's of crosses, and where the pedigree has been or can be investigated for some of the material for five to ten generations. What may happen in the case of plant hybrids, I am not able from personal observation to assert, but there is enough weight of evidence here to make one pause before one is prepared to admit that Mendel, or his followers, can change each conception of life in which heredity bears a part!

(3) Taking our stand then on the observed fact that a knowledge neither of parents nor of the whole ancestry will enable us to predict with certainty in a variety of important cases the character of the individual offspring we ask: What is the correct method of dealing with the problem of heredity in such cases? The causes $A, B, C, D, E, \ldots$ which we have as yet succeeded in isolating and defining are not always followed by the effect X, but by any one of the effects U, V, W, X, Y. We are therefore not dealing with causation but correlation, and there is therefore only one method of procedure possible; we must collect statistics of the frequency with which U, V, W, X, Y, Z respectively follow on $A, B, C, D, E\ldots$. From these statistics we know the most *probable* result of the causes A, B, C, D, E and the frequency of each deviation from this most probable result. The recognition that in the existing state of our knowledge the true method of approaching the problem of heredity is from the statistical side, and that the most that we can hope at present to do is to give the *probable* character of the offspring of a given ancestry, is one of the great services of Francis Galton to biometry.

* Mendelianism fails also for skin-colour in crosses between the black and white races of man.

† Other characters in pedigree stock are being taken into consideration at present.

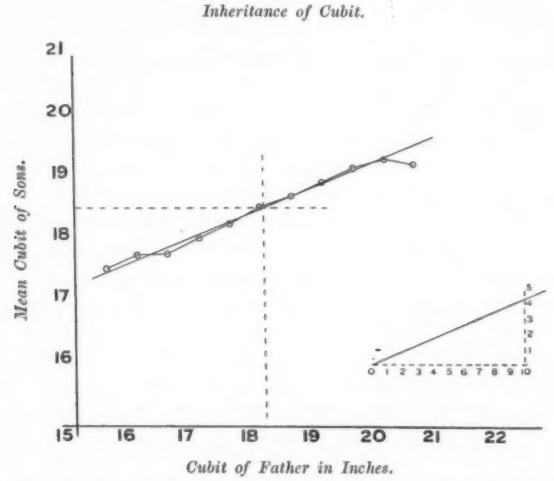
Yet since the publication of Francis Galton's *Natural Inheritance* in 1889 much progress has been made in the theory of multiple correlation, and also in our treatment of characters not quantitatively measurable, but which we may reasonably suppose to be based on quantitative factors,—as the degree of yellow pigment in the human eye. Further larger series of observations have been collected and reduced, and we are much clearer than we were ten years ago on the nature of regression and 'blend.'

Given two relatives P and Q the general theory of statistics tells us how to dress a table from which we can read off at once the most probable character of P when we know the character of Q. There is no biological theory in this, it is merely an arrangement of statistical information into classes. Such classes may be arranged on an actual quantitative scale, or as in the case of eye or hair colour on a qualitative scale, which with a high degree of probability corresponds to a true quantitative scale in some one or other pigment, whose amount cannot be actually determined. In both cases the methods of statistics allow us to determine:

(a) The variability in character of both P and Q in terms of the range of one of these arbitrary classes.

(b) The average variability in the same terms of an array of P's corresponding to a given Q, or of an array of Q's corresponding to a given P.

(c) The first two terms of the closest linear function which expresses the probable character of P or Q in terms of the known character of Q or P.



Probable value of cubit of son Q, for a known cubit in father P. A marked deviation from linearity only occurs in the extreme case of father's cubit, where there were only a few instances to base the son's mean upon.

All this has nothing in it peculiar to heredity, it is simply an application of the higher theory of statistics.

If we ask how closely the above linear function gives the probable Q-value from the known P-value; the answer is: Admirably, for those characters which have been actually tested. I have 21 tables involving three characters for diverse relationships in man and each embracing upwards of 1000 pairs. I choose one, the cubit, perfectly at random and it will be seen at once that, within the limits of random sampling, no function could give the probable character of Q for a given value of P with greater efficiency.

The line here dealt with is the so-called regression line; it is a purely statistical result, and has no relation to any biological theory or hypothesis as to heredity. Its determination depends on the so-called coefficient of correlation of P and Q, which we will write r_{pq}, and their variabilities σ_p and σ_q. In addition we must know the mean, the modal or, what is often sufficient, the median values of P and Q.

Now suppose that instead of one relative P we have any number P_1, P_2, P_3..., and let the corresponding statistical constants be r_{p_1q}, r_{p_2q}, r_{p_3q} ... σ_{p_1}, σ_{p_2}, σ_{p_3}.... Then the theory of statistics shows us that, if p_q be the most probable deviation of Q from the type of its generation, and h_{p_1}, h_{p_2}, h_{p_3}... be the observed deviations of P_1, P_2, P_3... from the types of their respective generations, then:

$$p_q = J_1 \frac{\sigma_q}{\sigma_{p_1}} h_{p_1} + J_2 \frac{\sigma_q}{\sigma_{p_2}} h_{p_2} + J_3 \frac{\sigma_q}{\sigma_{p_3}} h_{p_3} \quad \text{.....(i),}$$

where J_1, J_2, J_3... are known expressions involving only the r_{pq}'s, and can be calculated as soon as the latter have been found from observation.

Further the variability of Q about its most probable value, when we know the P's, takes on the average the value $\sigma_q \times \Delta$, where Δ is another well-known and calculable function of the r_{pq}'s.

The above is in no sense a biological theory, it is based on no data whatever except the actual statistics; it is merely a convenient statistical method of expressing the observed facts. If the facts are there it expresses them up to a certain point,—the most probable or most frequent value of the individual given his relatives. It cannot possibly within this range be upset by any hypothesis of heredity, Mendelian or otherwise, for it is based on no biological assumptions whatever. It is merely a convenient description of statistical tabulations.

I now proceed to limitations based on actual experience.

(i) The correlation coefficients between relatives are positive.

There is overwhelming evidence in favour of this in insects, animals and plants, for direct relationship back to great-great-grandparents and in a considerable range of collateral relationships—fraternal and avuncular.

But it may be shewn that if the r's are positive, increasing the number of relatives used as a basis of prediction reduces the value of Δ. The rate of reduction decreases considerably as we increase the number of correlations, but for the first increase from one on to two, three ... six, etc. it is very sensible.

We therefore conclude that:

There will be less average deviation from the probable character of an individual when we increase the number of relatives on which the prediction is based.

As a special case we say that:

Prediction will be closer when we use the ancestry as well as the parents in forming it, than when we use the parents alone.

As soon as we accept the positive character of the correlation coefficients between offspring and ancestry this is an absolutely certain result. It could only be invalidated if the parents defined the offspring absolutely, i.e. if Δ for two parents was zero, or the correlation with parentage perfect. But this is completely negatived by the fact * that the same pair of parents will produce offspring with very different characters,—e.g. two bay horses may give a black horse and not only a bay. Thus it seems to me that any hypothesis of inheritance which neglects ancestry is foredoomed to failure in the wide range of vital types for which we have shewn already that parentage does not define offspring uniquely. The variation in offspring of the same parents is not merely confined to slight differences in size of organs, but in many covers differences in colour, for example, as great as those upon which the Mendelians have based their discrimination between dominant and recessive characters.

(ii) In actual measurement on many series the J's come out less than unity and there are theoretical reasons for holding that this must always be the case.

Hence if the variability of the offspring generation be not much larger than that of the generation of any relative, the probable deviation from type of any individual will only receive from any relative a fraction of his observed deviation from his type.

This is Mr Galton's principle of regression, the exceptionality of P_1 is on the average only exhibited in part by his relative Q. This purely statistical and legitimate conclusion was seized upon as a biological law, and all life, but for constant selection, was assumed to be in a state of regression to some distant ancestry. The expression (i) does not warrant this assertion at all. To begin with there is no reason to suppose:

(a) That $\sigma_q = \sigma_{p_1} = \sigma_{p_2} = \sigma_{p_3} = \ldots$ or that the variability of all generations is the same. In numerous and large series I have not found it so†.

* For all the characters yet dealt with whether quantitative or qualitative in insects, animals and plants, there is no approach to $\Delta = 0$.

† The offspring generation tends almost universally to be more variable than that of any ancestral generation—parents are a selected portion of the community.

(*b*) $h_{p_1}, h_{p_2}, h_{p_3} \ldots$ are deviations from the type of each generation. I have never found the types of each generation identical. They often differ very sensibly.

(*c*) Each individual ancestor contributes only a fraction of his deviation from type to the probable deviation in type of the offspring. But if the ancestry be maintained for two or three generations at a given deviation from type the contributions of the different terms of (i) provide a probable deviation in type of offspring which may be equal to or may even exceed that of the ancestry.

This point is over and over again forgotten when biologists talk of regression as if it were a persistent retrogressive factor.

As a matter of fact with the numbers we already have for man, breeding true for three or four generations gives a value of p_q within a small percentage of the selected h_p, and this is only slowly modified, if the stock continues to breed true to itself. There is no such thing in statistical theory as a *necessary* regression if the selected stock pair with selected stock. The rapid establishment of breeds is not evidence therefore against the present view of heredity. On the contrary it flows at once from it.

(4) So far there is practically no assumption in our treatment of heredity, which has not been justified by ample experience, e.g. the close linearity in distribution of the probable value of a character in one relative for a known value of a character in a second. Further it enables us to predict probable values from any group of known relatives. When however we predict from direct ancestors only we state more particularly the law of ancestral heredity. If we are content with parents, or possibly in some cases with grandparents also, we have material for making a fairly close prediction, but if we want to deal with whole lines of ancestry, we are met at once by the difficulty of collecting statistical material for the correlation of the offspring's character with that of the higher antecedents. In man few observations or measurements have been made on a higher than the grandparental generation, and even in pedigree animals, where we can go much further back, the characters recorded are never quantitative, but concern colour or markings. To get over this absence of material Mr Galton originally proposed that we should correlate not with each individual ancestor but with the mean of each ancestral generation, females being reduced to a male standard. There is no assumption in this because, of course, correlation can statistically be worked for any such group of ancestry. But as we do not know the values of the correlation coefficients of the higher groups, Mr Galton suggested that we should take the $J_1, J_2, J_3, \ldots$ of equation (i) on p. 217 equal to $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$ respectively. This was undoubtedly an assumption, although not an unreasonable one *à priori*. That the intensity of ancestral heredity diminishes as we go backwards is demonstrated by both experiment and observation; and a geometrical series naturally first arises as a measure of such diminishing influence. But the law could only be demonstrated on the base of the first few terms of the *J*-series, and

Mr Galton's evidence from stature only gave a rough test of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ having the right trend. He did not himself claim more for it than this; thus after suggesting that $J_1 = \frac{1}{2}$ and $J_2 = \frac{1}{4}$, he writes: "It would, however, be hazardous to extend this sequence with confidence to more distant generations*."

Referring again to this point, Mr Galton says in 1897: "I stated it briefly and with hesitation in my book on *Natural Inheritance*†."

When I personally first came to investigate the matter from the more purely mathematical side, it seemed to me better to start with the coefficients of correlation and make no assumption as to the values of the regression coefficients in the J-series of p. 217‡. Each coefficient found would serve as the basis of a limited amount of prediction, and we must plod away finding coefficients for as many generations for as many characters in as many races as we possibly could. When these were known, we should possibly be able to predict the relationship between the correlation coefficients of successive generations even beyond the limits of observation and experiment. If we had to make an hypothesis at all, it seemed to me most reasonable to suppose the correlation not the regression coefficients of each generation of ancestry to diminish in a geometrical series. This view I kept steadily before me, but it was very difficult to find material going sufficiently far back to test it. Theoretically I shewed that if these correlations formed a geometrical series then the regression coefficients, or J's, would also form a geometrical series, if we neglected the effects of assortative mating§, but the data I had collected did not as far as they went justify Mr Galton's proposed $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$ series.

Taking the matter broadly we may say that the Law of Ancestral Heredity implied two ideas:

(*a*) That the proper method to proceed in heredity within the race is by the statistical theory of multiple correlation—this does not exclude the truth of any physiological theory, although it may serve to confirm or refute such a theory. The correlation between parent and offspring in man will remain about ·45 to ·5, whatever theory of gametes may eventually be accepted, and the prediction of probable character in son from actual character in father remains equally valid.

(*b*) That a knowledge of the nearer coefficients of correlation, i.e. those between offspring, and parents, grandparents, and possibly great-grandparents, will suggest the more distant ones, and that probably these, and consequently the multiple regression coefficients, are expressible as a geometrically decreasing series.

This is the hypothesis involved in my own expression of the Law of Ancestral

* *Natural Inheritance*, p. 136.

† *R. S. Proc.* Vol. XLI. p. 401. I have elsewhere shewn that the evidence from Basset Hounds admits of other interpretations. *R. S. Proc.* Vol. 66, p. 140 *et seq.*

‡ "Regression, Panmixia and Heredity." *Phil. Trans.* Vol. 187, p. 253 *et seq.*

§ "On the Law of Ancestral Heredity." *R. S. Proc.* Vol. 62, p. 394.

Heredity, and I believe it is a generalisation, which Mr Galton might, perhaps, be willing to accept in place of his much more definite series of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

(5) Assuming we had the data for quantitatively measurable characters one would prefer to deal with such, but failing this I have asked myself how far the successive correlation coefficients for ancestry in qualitative characters (which we may reasonably suppose to have a quantitative scale behind them) fall into a geometrical progression. In eye-colour for man I have previously given* the four parental and the eight grandparental correlations. Mr F. Lutz has completed† the series, as far as it is possible to do on Mr Galton's material, by investigating the correlation of offspring with the great-grandparents, arranged in four groups. Mr L. Bramley-Moore and I had given the four parental and two of the grandparental coat-colour correlations for horses in 1899‡. Mr N. Blanchard published two further grandparental correlations recently§, and he has now completed the whole series of eight, and they will be found in a note attached to this paper. Dr Alice Lee worked out the great-grandparental and the great-great-grandparental correlations from the data provided by Mr Blanchard's MS. pedigree books; these are given in a second note below.

We have reached the following results:

Mean Correlation	Eye-colour in Man	Coat-colour in Horse
Parental	·4947	·5216
Grandparental	·3166	·2976
Great-grandparental	·1879	·1922
Great-great-grandparental	—	·1469

When we note that the probable error of these coefficients is about the order ·03, one is at once struck with their substantial agreement. We are forced to the conclusions:

(i) That eye-colour in man and coat-colour in horses are inherited in sensibly the same manner.

(ii) That the original series proposed by Mr Galton $\frac{1}{3}$, $\frac{1}{9}$ ... for these correlations‖ will not fit them at all.

It might be supposed that pigmentation followed some other rule than measurable characters. But in recent work on measurable characters for 12 series, each numbering upwards of 1000 cases, I have found for man the mean parental value ·46. This is the largest inheritance series for man yet worked out, being based on measurements of upwards of 1100 families, so I am pretty

* *Phil. Trans.* A, Vol. 195, p. 106.

† See Note on the Influence of Change in Sex on Inheritance. *Biometrika*, Vol. II. p. 237.

‡ *Phil. Trans.* A, Vol. 195, p. 93.

§ *Biometrika*, Vol. I. p. 361.

‖ *Natural Inheritance*, p. 133.

confident that parental inheritance for man lies between ·45 and ·5, and is quite incompatible with a value of $\frac{1}{3}$*.

I now turn to the problem of how far the data can be represented by a geometrical series. I fitted the best geometrical series first to the horse only, then to man, and finally to both with the following results.

A. Horse, observed	Best Geometrical Series	Man, observed
·52	·48	·49
·30	·32	·32
·19	·21	·19
·15	·14	—

B. Man, observed	Best Geometrical Series	Horse, observed
·49	·50	·52
·32	·31	·30
·19	·19	·19
—	·12	·15

In the first case, the series being a, ar, ar^2, ... we have:

$$a = ·4809 \text{ and } r = ·6602.$$

In the second case:

$$a = ·5007 \text{ and } r = ·6167.$$

If we take the best geometrical series embracing the data for both races we have:

C. Horse, observed	Man, observed	Best Geometrical Series
·52	·49	·49
·30	·32	·32
·19	·19	·20
·15	—	·13

Here we have:

$$a = ·4921 \text{ and } r = ·6428.$$

Lastly, if we take $a = ·5$, $r = \frac{2}{3}$ as close round numbers, we have:

D. Horse, observed	Man, observed	Close Series
·52	·49	·50
·30	·32	·33
·19	·19	·22
·15	—	·15

Within the errors of observation this close series represents excellently the observed results for either eye-colour in man or coat-colour in the horse. In other words:

* As long as the only data for a measurable character in man—Mr Galton's stature data—gave a parental correlation of about $\frac{1}{3}$ it seemed needful to emphasise the distinction between the results for eye-colour and stature. In the light of my present knowledge, the distinction between the two classes of characters seems not so clear or needful. A large series of pedigree dogs at present in hand gives a parental correlation almost identical with that of horse and man.

As far as the available data at present go inheritance coefficients for ascending ancestry are within the limits of observational error represented by a geometrical series and by the same series.

From this it follows that*: *The contributions of the ancestry also follow a geometrical series, although not that originally proposed by Mr Galton.*

(6) Mr Galton has assumed, that if the relatives include all the ancestry, and if all these ancestry had the same deviation h, the offspring will have a probable deviation of h. This is really the introduction of a biological hypothesis, the truth of which can only be tested by observation. Mr Galton deduces this result in the following manner: he supposes a stable population, i.e. one in which the mean and variability of each generation remain the same, and the parents in each generation are the whole or at any rate a random sample of that generation; there must also be no reproductive selection, or fertility must not be correlated with the character of which the inheritance is under consideration. Further there must be no assortative mating. Under these circumstances we have the following form of (i) p. 217:

$$p_q = (J_1 + J_2 + J_3 + \ldots)\, h,$$

and since Mr Galton holds that p_q will then equal h, we have

$$J_1 + J_2 + J_3 + \ldots = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Any geometrical series $\epsilon(1 + \rho + \rho^2 + \ldots)$ for the J's will satisfy this condition if $\epsilon = 1 - \rho$; the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is not the only one satisfying (2)†.

But it is doubtful‡ how far this conclusion is justifiable. Statistically it is of course unnecessary. (i) is a relation between the probable deviation of Q from its mean and the actual deviations of each P from their individual means. There is no reason why the means of Q and of all the P's should be the same. They may be different owing to environment or to selection. Further there is no reason why the variabilities should be the same; parents may be a selection out of the general community in each generation. As soon as we realise that the h's are deviations from the generation means, and these are not all the same, and that the variability in each generation differs, the need for the relation

$$J_1 + J_2 + J_3 + \ldots = 1$$

ceases to be apparent.

In data like eye-colour in man and coat-colour in thoroughbred horses there has been undoubtedly a secular change going on; the proportion of blue eyes in the latest offspring is considerably less than in the great-great-grandparents, while the early grey horses have largely disappeared from the stud-book. Further we find each ancestral generation is roughly speaking less variable than

* *R. S. Proc.* Vol. 62, p. 394.

† *R. S. Proc.* Vol. 62, p. 402, and Vol. 66, p. 147.

‡ It is hardly consonant, for example, with what we know of eye-colour, where there is an assortative mating coefficient of ·1002: see *Phil. Trans.* A, Vol. 195, p. 113.

the one below it. Men who live to be parents are a more select group than their offspring and this is still more true of grandparents and higher ancestry. To be a great-great-grandparent means to have a healthy stock, and such men form a selection only of the community in each generation. Hence neither for man nor the thoroughbred horse need we *à priori* expect to find, for the correlation coefficients as actually determined, the relation $J_1 + J_2 + J_3 + \ldots = 1$ or $\epsilon = 1 - \rho$ satisfied. The actual statistics for man and the horse conclusively prove that it is not satisfied. If we had a population in which all members were randomly mated and were parents of equal fertility; further if every generation had the same mean and variability, then we might have $\epsilon = 1 - \rho$, but then the correlations of ancestry and offspring would probably take very different values from what we find them to have in the cases of man and horse under existing circumstances.

(7) Supposing all mating to be random * we can find ϵ and ρ and all the J's by the method of my first paper on the "Law of Ancestral Heredity†." In the notation of the present paper, if ar^n be the correlation coefficient of "the $n+1$th mid-parent" with the offspring, and if $J_q = \epsilon' \rho'^{q-1}$, we find:

$$r = \rho' + \epsilon' \quad \ldots\ldots(3),$$

$$-a + \epsilon' \rho' a/(1 - \rho' r) = -\epsilon' \quad \ldots\ldots(4);$$

whence:

$$\rho'^2 - \rho' \frac{1 + r^2 - 2ar}{r - a} + 1 = 0 \quad \ldots\ldots(5).$$

Hence taking $a = \cdot 5$ and $r = \frac{2}{3}$ as close enough to the observed values we have:

$$\rho' = \cdot 2251, \qquad \epsilon' = \cdot 4415.$$

Formula (i) of p. 217 using "mid-parents" now becomes:

$$p_q = \cdot 4415 \left\{ \frac{\sigma_0}{\Sigma_1} h_{p_1} + \cdot 2251 \frac{\sigma_0}{\Sigma_2} h_{p_2} + (\cdot 2251)^2 \frac{\sigma_0}{\Sigma_3} h_{p_3} + \ldots \right\}.$$

Here Σ_1, Σ_2, Σ_3 ... are the standard deviations of the mid-parental groups, and $\Sigma_s = \sigma_s/(\sqrt{2})^s$, if σ_s be the standard deviation of the sth generation of ancestors. Thus:

$$p_q = \cdot 4415 \times \sqrt{2} \left\{ \frac{\sigma_0}{\sigma_1} h_{p_1} + \cdot 2251 \sqrt{2} \frac{\sigma_0}{\sigma_2} h_{p_2} + (\cdot 2251 \sqrt{2})^2 \frac{\sigma_0}{\sigma_3} h_{p_3} + \ldots \right\}$$

$$= \cdot 6244 \left\{ \frac{\sigma_0}{\sigma_1} h_{p_1} + \cdot 3184 \frac{\sigma_0}{\sigma_2} h_{p_2} + (\cdot 3184)^2 \frac{\sigma_0}{\sigma_3} h_{p_3} + \ldots \right\}$$

$$= \epsilon \left\{ \frac{\sigma_0}{\sigma_1} h_{p_1} + \rho \frac{\sigma_0}{\sigma_2} h_{p_2} + \rho^2 \frac{\sigma_0}{\sigma_3} h_{p_3} + \ldots \right\}.$$

Clearly $\epsilon = \cdot 6244$ is not equal to $1 - \rho$ or $\cdot 6816$, or Mr Galton's hypothesis is certainly not satisfied either for man or horse. But since we know in these cases

* *R. S. Proc.* Vol. 66, p. 149.

† *R. S. Proc.* Vol. 62, p. 394.

that $\sigma_1 = \sigma_2 = \sigma_3 = \ldots = \sigma_0$ is not true, and further that $h_{p_1}, h_{p_2}, h_{p_3} \ldots$ are measured from different means, we have no *à priori* right to suppose it ought to be satisfied.

(8) It is not without interest to follow up what would happen in man and horse if, with our present numbers, we supposed the conditions for a stable population to be really satisfied. Let us select about the deviation h from the common mean for s generations, then the offspring would have a probable deviation from that mean of

$$\epsilon \frac{(1-\rho^s)}{1-\rho} h.$$

If we then ceased to select, but bred the offspring among themselves for t generations, the offspring of the tth generation would have a probable deviation from the mean of

$$\epsilon \cdot \frac{1-\rho^s}{1-\rho} \cdot (\sqrt{2}r)^t h.$$

Let us turn these results into numbers, remembering that $\epsilon = \cdot 6244$ and $\rho = \cdot 3184$.

Result of selecting all ancestry for one	generation			$= \cdot 62h$
" " "	two	generations		$= \cdot 82h$
" " "	three	"		$= \cdot 89h$
" " "	innumerable	"		$= \cdot 92h$.

In other words indefinitely long selection would only raise us to ·92 of the desired character. We should therefore not get the whole of the desired character *without selecting beyond it.* But on the other hand by merely selecting for two generations we should get within ten, and for three generations within four per cent. of the possible maximum of indefinitely protracted selection. Hence selection for a very few generations would raise the selection rapidly to within a small percentage of the maximum result.

Generations after which selection is stopped and simple breeding-in of selected stock follows.

	First	Second	Third	Innumerable
Last offspring of selection ...	·62h	·82h	·89h	·92h
First generation of in-bred stock	·59h	·78h	·84h	·86h
Second " " "	·56h	·73h	·79h	·81h
Third " " "	·52h	·67h	·74h	·77h
Tenth " " "	·35h	·46h	·49h	·51h

Next let us suppose selection stopped and the stock to breed-in or mate with its likes. The above table shows how it would slowly degenerate. The essential

point to be noted is that while two or three generations of selection would carry us up to 80 or 90 per cent. of the desired character, ten generations following this of merely in-breeding without any selection would not have cost us 50 per cent. of the character so acquired. Quick influence of selection, slow effect of regression would be the result of combining the actually observed values with Mr Galton's theory as to what should hold for a stable population.

The reader must not forget that the illustration here given is absolutely hypothetical; the statistical constants obtained are deduced from material to which Mr Galton's conditions hardly apply even as a rough approximation. Yet it is possible that something of the kind here indicated may occur in special cases. But if so, we ought to be very cautious of using vague categories in problems of heredity. If the mean tint of a seed, say, be yellow, and h would carry us well into the green end of the scale, $\cdot 4h$ might still be green, and certainly for three and possibly for a good many more generations we might consider the stock arising from a *single* selection to be breeding true to itself, although actually it might be slowly regressing to the original tint of the early ancestry. It seems absolutely necessary in all such cases to have some colour standard and determine quantitatively whether successive generations do or do not tend to slowly approach or depart from it. The statement that ancestry has no influence might well be deduced by the use of a rough category, which would still class $\cdot 62h$ with $\cdot 52h$.

(9) *Conclusions.*

(*a*) In all cases as those of man, horse and dog, where parents of identical character do not produce identical offspring, the theory of statistics shows us that closer prediction may be obtained when we predict from many instead of few relatives. This follows from the consideration that all the heredity coefficients are *positive.*

Attention is therefore properly paid to ancestry in such cases, and it is very misleading to suggest that any law of heredity can be universal which neglects ancestry.

(*b*) The law of ancestral heredity in its most general form is not a biological hypothesis at all, it is simply the statement of a fundamental theorem in the statistical theory of multiple correlation applied to a particular type of statistics. If statistics of heredity are themselves sound the results deduced from this theorem will remain true whatever biological theory of heredity be propounded.

(*c*) The law of ancestral heredity as founded on the theory of multiple correlation involves no *biological* theory of regression. The term regression has unfortunately been taken from statistical theory and interpreted in a biological sense. In statistics the regression is always to the mean of the forecasted character, but no assumption is made that this mean is identical with that of the foreknown character*. Further, if there be a number of cognates, we can *à priori*,

* There is a "regression" for example if we predict breadth of skull from its length. I think it might be useful to adopt the word "predicate" for the forecasted and "cognate" for the foreknown character.

i.e. before quantitative analysis, not state whether the total amounts they will contribute to the predicate will or will not indicate a biological regression*.

(*d*) The law of ancestral heredity as a pure statistical statement has been supplemented by hypotheses which need verification and are semi-biological. These may (i) either be hypotheses as to the multiple regression coefficients, or (ii) as to the individual coefficients of correlation.

Mr Galton has assumed that the former are the geometrical series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$. I have assumed that the *mean* correlation coefficients for each ancestral generation form some geometrical series. It follows from my assumption that the regression coefficients would also form a geometrical series, but not necessarily Mr Galton's.

(*e*) In eye-colour in man and coat-colour in horses the mean ancestral coefficients of regression form within the limits of errors of random sampling a geometrical series, but it is not Mr Galton's series:

$$\cdot 5000, \quad \cdot 2500, \quad \cdot 1250, \quad \cdot 0625 \ldots,$$

but more nearly:

$$\cdot 6244, \quad \cdot 1988, \quad \cdot 0630, \quad \cdot 0202 \ldots.$$

In other words actual statistics show that in man and horse the parents are much more and the grandparents and higher ancestry less influential than on Mr Galton's hypothesis.

Thus the law of ancestral heredity (by which we are to understand the theory of multiple correlation together with the hypothesis that the mean ancestral correlations or the regression coefficients form a geometrical series) fits the data for horse and man remarkably well.

(*f*) In man and horse we find the means of each generation differ, and further the variabilities of each generation differ. It is an assumption to suppose under these circumstances that the sum of the regression coefficients (or rather that part of them which we have represented by $J_1, J_2, J_3, \ldots$) is unity.

Any geometrical series for the regression coefficients which satisfies the condition $\epsilon = 1 - \rho$ (like Mr Galton's does) would give on the supposition of equal means and variabilities for each generation *no regression whatever* after a stock began to in-breed.

If we may apply (which is very doubtful) our values for the J's in man and horse to cases in which the means and variabilities of each generation remained the same, there would result the following principle:

Two or three generations of selection would produce a stock of upwards of 90 per cent. of the selected character, but no amount of selection, unless of a greater

* It is curious that the original numbers selected by Mr Galton for the regression coefficients $\frac{1}{4}, \frac{1}{2}, \frac{1}{3}$, etc. indicate no regression whatever towards the predicate mean, after the first generation, if the stock in-breeds or breeds with its likes. It is characteristic of how conceptions are misunderstood, that this point of "regression" is what the majority of biologists have seized as the one easily comprehended principle out of the whole of Mr Galton's work!

than the desired amount of character, would give us more than 92 per cent. of it. After selection ceases a very slow regression sets in, which would be hardly perceptible without very definite quantitative measurement for the first three or four generations of in-breeding.

No rough classification like a Mendelian category would enable us to test whether such slow regression is really taking place, i.e. whether "recessives" are really breeding true to their stock*. And above all no Mendelian theory can (*a*) replace the statistical treatment unless it shows that offspring are absolutely determined by their parents, for if the correlation be not perfect, a better prediction must be obtainable by using the positively correlated ancestry, or (*b*) be proved without careful quantitative analysis of the variations falling within the wide categories adopted.

Appendix I. Note on a paper by G. U. Yule.

This paper was written before I was aware that Mr G. U. Yule was writing some account of the law of ancestral heredity, and was sent to press before I had seen his paper (*The New Phytologist*, November and December, 1902). With much of his paper, I agree, for example, with his insistence on the point that the laws of intra-racial heredity are not incompatible with Mendelian principles holding for hybridisation. I should lay, however, far less stress than he does on the value of the existing evidence in favour of those principles. That evidence certainly justifies the making of new and crucial experiments, but these will have to be made with much greater caution and closer quantitative definition of the categories employed before we can say with certainty how far either of the two principles of dominance and segregation actually applies even to the cases of hybridisation already dealt with by the Mendelians. I shall be surprised if the laws which govern intra-racial and inter-racial heredity are not more closely related than appears to me possible under the recent formulation of Mendelian Principles by Mr Bateson. Further Mr Yule does not seem to me to have clearly expressed my personal position with regard to the law of ancestral heredity. I believe that my memoir of 1896 was the first in which the equations of multiple regression were worked out and applied to the problem of heredity. Such equations are, I presume, what Mr Yule refers to and says may be termed the law of ancestral heredity. But the objection to such form of the law is simply the impossibility of working it out for all the ancestry, and for all characters of all races. The problems that arise at once in our minds are: Is there any relationship between the correlations of offspring and successive ancestors? Are these correlations in cases, where direct selection of the characters is small or can be allowed for, the same or approximately the same for different characters of the same race, or for divers characters of different races? Is there any relation between the correlations, which appears fairly in accordance with observation and yet will simplify the otherwise appalling complexity of the multiple regression equations? The memoir of 1896 gave the general regression equations, and indicated what results would flow supposing one or other of a variety of hypotheses due to Galton, Weismann and others were true. The memoir of 1898 adopted the simpler hypothesis that the correlation coefficients decrease in geometrical progression, *it did not involve the fixity of the numerical constants of heredity* which Mr Yule tells us has not

* For example, if we selected extreme chestnut (towards roan-chestnuts) for two generations only in the horse, yet if the progeny in-bred for 10 generations, the offspring would still be well in the chestnut range of coat-colour, and no category like chestnut would tell us whether the horses were breeding true or regressing.

stood the test of time. This simpler hypothesis, discussed in the present paper, still seems to me to stand the test of time. Lastly, I think, the reader of Mr Yule's paper would believe from the words: "Selective mating, natural selection, reproductive selection, the effect of circumstance, had all in turn to be recognised as causes affecting the values of the constants of heredity," that these factors were introduced *post hoc*, i.e. after observation had failed to give fixity of constants, whereas almost the reverse is the true case. The definition of selective mating was given in my first paper to the Royal Society on biparental inheritance in 1895* before I had applied the formulæ to actual data at all. Reproductive selection was considered in 1896†, and the influence of natural selection on correlation (including the coefficients of heredity) was given in College Lectures of the session 1896–7, and put into a form identical with that of the memoir just issued‡ in August 1897. The memoir of 1898 on the law of ancestral heredity was written with a knowledge of the influence of the three factors of selective mating, natural selection and reproductive selection on the constants of heredity. As a matter of fact they had been biometrically defined in or before the memoir of 1896, and were not "all in turn recognised as causes affecting the values of the constants of heredity," or used to bolster up some theory which had not stood the test of time.

What I felt in 1896, I still feel, namely: that the problem of heredity from the purely statistical standpoint is an extremely difficult one, that the factors of selective mating, natural selection, reproductive selection, environmental influence are extremely complex, especially when we place plants or insects in very artificial surroundings for the purpose of experimenting on their laws of reproduction. Still these factors do not invariably tend in one direction, and when we take long series, as free from their influence as is practically possible, we shall be able to judge from the clustering of the heredity coefficients round certain values—a phenomenon which I think is obvious in existing results—what in broad lines is the quantitative intensity of heredity for different relationships, and for different characters in different species. Mr Yule indicates that he has a theory which allows him to determine the influence of the environmental, reproductive, selective and mating factors; I can only say that I, as well as most biometricians, will heartily welcome its publication, if it not only allows, in the case of any heredity statistics which it is feasible to collect, for the correcting of the heredity constants for all these influences, but achieves this by some hypothesis which is *à priori* as reasonable, and *à posteriori* as justified as the simple one that ancestral correlations diminish in a geometrical progression.

Appendix II. On Inheritance (Grandparent and Offspring) in Thoroughbred Racehorses.

By NORMAN BLANCHARD, B.A.

(1) At the suggestion of Professor Karl Pearson I have recently worked out the remaining four cases for the inheritance of coat-colour in thoroughbred horses, viz. those between foals—colts or fillies—and their paternal and maternal grandams. The work was done in a somewhat different manner to that on the four earlier tables. In order to consider the bearing of coat-colour inheritance on Mendel's theory, actual colour pedigrees were formed going back to the great-great-grandparents and in some cases to the fifth and eighth generation of ancestry. Further, in order to get more striking colour contrasts than are now-a-days possible, the first three volumes of the stud-book were dealt with, I thus went back to the earliest records of coat colour in thoroughbreds. Here we find 2 to 3 per cent. of black horses and 9 to 10 per cent. of

* "On Regression and Inheritance in the case of two Parents," *R. S. Proc.* Vol. 58, 1895.

† "On Reproductive Selection," *R. S. Proc.* Vol. 59, p. 301, 1896.

‡ *Phil. Trans.* Vol. 200, pp. 1—66, 1902.

greys, while in the recent studbooks these percentages are hardly 2 per cent. and ·5 to 1 per cent. respectively. In the early colour records the finer distinctions, e.g. "black or brown," "bay or chestnut," etc., of the modern terminology rarely appear, but this is not of importance. A thousand cases were taken for each series, and the correlation Tables I. to IV. were constructed as in Pearson and Bramley-Moore's memoir on *Inheritance of Coat-Colour in Horses**, in fact the method there used was followed identically, the fourfold division between chestnut and bay being used for the short tables. The following results were obtained, the notation being that of Professor Pearson's memoir: *On the Correlation of Characters not quantitatively measurable*†.

(2) Table I. reduces to

Paternal Grandams.

Colts.	Bay and darker	Chestnut and lighter	Totals
Bay and darker ...	486	188	674
Chestnut and lighter	172	154	326
Totals	658	342	1000

This gives:

$$h = \cdot 4070105, \quad H = \cdot 3672283,$$
$$k = \cdot 4509851, \quad K = \cdot 3603667,$$

and the equation:

$$\cdot 3212093 = r + \cdot 0917778r^2 + \cdot 1107746r^3 + \cdot 0606283r^4 + \cdot 0308582r^5 + \cdot 0443421r^6 + \cdot 0101909r^7 + \cdot 0339243r^8 + \text{etc.}$$

the root of which is

$$r = \cdot 3085.$$

From Table II. we have

Maternal Grandams.

Colts.	Bay and darker	Chestnut and lighter	Totals
Bay and darker ...	539	149	688
Chestnut and lighter	198	114	312
Totals	737	263	1000

giving:

$$h = \cdot 6341100, \quad H = \cdot 3300624,$$
$$k = \cdot 4901880, \quad K = \cdot 3537792,$$

* *Phil. Trans.* A, Vol. 195, pp. 92 *et seq.*
† *Phil. Trans.* A, Vol. 195, pp. 1—47.

and the equation:

$$\cdot 2735648 = r + \cdot 1554165r^2 + \cdot 0631895r^3 + \cdot 0928545r^4 + \cdot 0100882r^5 + \cdot 0608670r^6 - \cdot 0007343r^7 + \cdot 0412678r^8 + \text{etc.}$$

the root of which is

$$r = \cdot 2614.$$

From Table III.

Paternal Grandams.

Fillies.	Bay and darker	Chestnut and lighter	Totals
Bay and darker ...	525	189	714
Chestnut and lighter	175	111	286
Totals	700	300	1000

giving:

$$h = \cdot 5243986, \quad H = \cdot 3517190,$$
$$k = \cdot 5689150, \quad K = \cdot 3393338,$$

and the equation:

$$\cdot 2111434 = r + \cdot 1491612r^2 + \cdot 0817246r^3 + \cdot 0906581r^4 + \cdot 0138143r^5 + \cdot 0606133r^6 + \cdot 0014633r^7 + \cdot 0420798r^8 + \text{etc.}$$

the root of which is

$$r = \cdot 2041.$$

From Table IV.

Maternal Grandams.

Fillies.	Bay and darker	Chestnut and lighter	Totals
Bay and darker ...	542	163	705
Chestnut and lighter	187	108	295
Totals	729	271	1000

giving:

$$h = \cdot 6097900, \quad H = \cdot 3312572,$$
$$k = \cdot 5388334, \quad K = \cdot 3380886,$$

and the equation:

$$\cdot 2505039 = r + \cdot 1750152r^2 + \cdot 0701968r^3 + \cdot 0976633r^4 + \cdot 0088602r^5 + \cdot 0630885r^6 + \cdot 0001216r^7 + \cdot 0419921r^8 + \text{etc.}$$

the root of which is

$$r = \cdot 2392.$$

(3) The eight grandparental correlations for the horse are given below together with the corresponding ones for eye-colour in man for the sake of comparison. The whole series has now been determined by Pearson and Bramley-Moore* or myself†.

* *Loc. cit.* p. 93.

† *Biometrika*, Vol. I. p. 361.

		Horse. Coat-colour	Man. Eye-colour
1	Pat. Grandfather and Son ...	·3238	·4213
2	Pat. Grandfather and Daughter	·3609	·3802
3	Mat. Grandfather and Son ...	·3590	·3717
4	Mat. Grandfather and Daughter	3116	·2969
5	Pat. Grandmother and Son ...	·3085	·2722
6	Pat. Grandmother and Daughter	·2041	·2205
7	Mat. Grandmother and Son ...	·2614	·2623
8	Mat. Grandmother and Daughter	·2392	·3180
	Mean	·2961	·3179

It will be seen at once that the *mean* result for men is substantially in agreement with that for horses. We can hardly suppose grandparental correlation for the characters in man and horse dealt with to differ much from ·3. The mean value of grandparental correlation in the horse for cases 1 to 4 drawn from the modern studbooks is ·3388 and for cases 5 to 8 drawn from the older records ·2533. I believe this is due to the close relationship of so many of the horses in the earliest period of breeding. The difference may also to some extent depend on the somewhat rougher appreciation and description of colour. If there be any influence of change of sex in the intensity of heredity for coat-colour in the horse, it is not manifested in the above statistics, which thus differ widely from those of eye-colour in man.

TABLE I.

Colts and Paternal Grandams.

Paternal Grandams.

Colts.	bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	Totals
bl.	8	—	—	6	—	—	11	—	—	12	—	—	—	—	—	3	40
bl./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
br./bl.	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1	2
br.	8	—	—	39	—	—	75	—	—	20	—	2	—	—	—	7	151
br./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b.	13	—	—	95	—	—	230	—	—	101	—	2	1	—	—	39	481
b./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch.	2	—	—	27	—	—	113	—	—	82	—	1	—	—	—	16	241
ch./ro.	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
ro./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro.	—	—	—	2	—	—	2	—	—	1	—	—	—	—	—	2	7
ro./gr.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
gr.	—	—	—	10	—	—	16	—	—	13	—	—	—	—	—	37	76
Totals	31	—	—	180	—	—	447	—	—	230	—	5	1	—	—	106	1000

TABLE II.

Colts and Maternal Grandams.

Maternal Grandams.

Colts.	bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	Totals
bl.	—	—	—	6	—	—	13	—	—	4	—	—	—	—	—	3	31
bl./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
br./bl.	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	2
br.	3	—	—	34	—	—	89	—	—	17	—	1	1	—	—	10	155
br./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b.	6	—	—	73	—	—	309	—	—	76	—	1	—	—	—	35	500
b./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch.	3	—	—	29	—	—	132	—	—	58	—	—	2	—	—	14	238
ch./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro.	—	—	—	1	—	—	1	—	—	1	—	—	1	—	—	1	5
ro./gr.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr./ro.	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
gr.	2	—	—	7	—	—	22	—	—	7	—	—	—	—	—	30	68
Totals	14	—	—	150	—	—	573	—	—	164	—	2	4	—	—	93	1000

TABLE III.

Fillies and Paternal Grandams.

Paternal Grandams.

Fillies.	bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	Totals
bl.	—	—	—	5	—	—	13	—	—	9	—	—	—	—	—	1	28
bl./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
br./bl.	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
br.	8	—	—	38	—	—	83	—	—	31	—	—	—	—	—	6	166
br./b.	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
b./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b.	15	—	—	91	—	—	270	—	—	109	—	—	—	—	—	32	517
b./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch.	2	—	—	25	—	—	113	—	—	60	—	—	—	—	—	14	214
ch./ro.	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
ro./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro.	—	—	—	1	—	—	4	—	—	—	—	—	—	—	—	—	5
ro./gr.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr.	1	—	—	12	—	—	17	—	—	11	—	—	—	—	—	25	66
Totals	26	—	—	172	—	—	502	—	—	221	—	—	—	—	—	79	1000

TABLE IV.

Fillies and Maternal Grandams.

Maternal Grandams.

Fillies.	bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	Totals
bl.	1	—	—	4	—	—	9	—	—	5	—	—	—	—	—	6	25
bl./br.	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
br./bl.	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
br.	1	—	—	33	1	—	84	—	—	29	—	—	1	—	—	7	156
br./b.	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
b./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b.	6	—	—	80	—	—	322	—	—	73	—	—	—	—	—	40	521
b./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch.	2	—	—	17	—	—	138	—	—	60	—	—	—	—	—	15	232
ch./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro.	—	—	—	1	—	—	2	—	—	—	—	—	—	—	—	—	3
ro./gr.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr.	1	—	—	9	—	—	17	—	—	8	—	—	—	—	—	25	60
Totals	11	—	—	144	1	—	573	—	—	177	—	—	1	—	—	93	1000

Appendix III. On Inheritance (Great-grandparents and Great-great-grandparents and Offspring) in Thoroughbred Racehorses.

By ALICE LEE, D.Sc.

From Mr Blanchard's racehorse coat-colour pedigrees, I have, paying no attention to sex, been able to extract 1155 cases of great-grandparent and offspring and 978 cases of great-great-grandparent and offspring. When it is noted that there are 16 types of great-grandparental and 32 types of great-great-grandparental relationship, so that 48 correlation tables would be required for the full working out of these cases, it will be noted why in this preliminary study, I have not differentiated between the sexes.

Tables I. and II. reproduce my data.

From Table I. I find in the fourfold division hitherto adopted:

Great-Grandparents.

Offspring.	Bay and darker	Chestnut and lighter	Totals
Bay and darker ...	601	264	865
Chestnut and lighter	166	124	290
Totals	767	388	1155

Hence

$$h = \cdot 671{,}102, \quad k = \cdot 423{,}602,$$

and the equation is:

$$\cdot 198{,}115 = r + \cdot 141{,}813 r^2 + \cdot 075{,}166 r^3 + \cdot 085{,}182 r^4 + \text{etc.}$$

giving as root

$$r = \cdot 1922.$$

From Table II. we have:

Great-great-Grandparents.

Offspring.	Bay and darker	Chestnut and lighter	Totals
Bay and darker ...	497	252	749
Chestnut and lighter	130	99	229
Totals	627	351	978

I find

$$h = \cdot 725{,}258, \quad k = \cdot 361{,}420,$$

and the equation

$$\cdot 149{,}992 = r + \cdot 131{,}061 r^2 + \cdot 068{,}681 r^3 + \cdot 077{,}532 r^4 + \text{etc.}$$

which gives

$$r = \cdot 1469.$$

We see from these results that the ancestral relationships in coat-colour for the horse are far more intense than was originally supposed for the like grades of relationship. We have in round numbers:

Parental correlation ... ... = ·50.
Grandparental correlation ... ... = ·30.
Great-grandparental correlation ... = ·20.
Great-great-grandparental correlation = ·15.

As far as this series goes, not the coefficient itself, but the difference appears halved at each ascent, and it is clear that the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ... series cannot possibly hold for the regression coefficients of the multiple correlation series.

TABLE I.

Great-Grandparents and Offspring.

Great-Grandparents.

Offspring.	bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	Totals
bl.	4	—	—	17	—	—	46	—	—	27	—	—	—	—	—	2	96
bl./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
br./bl.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
br.	6	—	—	35	—	—	97	—	—	49	—	—	—	—	—	9	196
br./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b.	17	—	—	82	—	—	297	—	—	149	—	—	—	—	—	28	573
b./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch.	1	—	—	34	—	—	103	—	—	75	—	—	—	—	—	22	235
ch./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro.	—	—	—	2	—	—	8	—	—	4	—	—	—	—	—	2	16
ro./gr.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr.	—	—	—	5	—	—	13	—	—	13	—	—	—	—	—	8	39
Totals	28	—	—	175	—	—	564	—	—	317	—	—	—	—	—	71	1155

TABLE II.

Great-great-Grandparents and Offspring.

Great-great-Grandparents.

Offspring.	bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	Totals
bl.	2	—	—	14	—	—	40	—	—	24	—	—	—	—	—	7	87
bl./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
br./bl.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
br.	3	—	—	21	—	—	69	—	—	43	—	—	—	—	—	6	142
br./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b./br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b.	5	—	—	70	—	—	273	—	—	143	—	—	—	—	—	29	520
b./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch./b.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ch.	1	—	—	17	—	—	89	—	—	61	—	—	—	—	—	19	187
ch./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro./ch.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
ro.	—	—	—	2	—	—	7	—	—	5	—	—	—	—	—	1	15
ro./gr.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr./ro.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
gr.	—	—	—	6	—	—	8	—	—	10	—	—	—	—	—	3	27
Totals	11	—	—	130	—	—	486	—	—	286	—	—	—	—	—	65	978

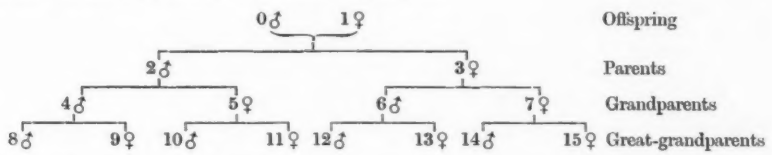
MISCELLANEA.

Note on the Influence of Change in Sex on the Intensity of Heredity.

By FRANK E. LUTZ, M.A.

(1) In Professor Karl Pearson's memoir, *On the Inheritance of Eye Colour in Man** it is stated on page 117 "that the younger generation is more highly correlated with an ascendant or collateral of the same than of the opposite sex," and again page 115, "that change of sex weakens the intensity of heredity." It seemed worth while pursuing this point further and grouping together all the available evidence as far as eye-colour in man is concerned. Accordingly, Mr Galton having again most kindly put his eye-colour data at Prof. Pearson's disposal, I have worked out the correlation between great-grandparents and offspring, which completes as far as ancestral correlation is concerned the whole of the available records of this sort. I hope shortly to examine in the same way a large mass of data which I have gathered concerning coat-colour in dogs of the Pointer breed.

I use the following scheme of notation for pedigree:



This only differs from Mr Galton's widely accepted scheme in the introduction of 0 for the male offspring to distinguish them from 1, the female offspring. $r_{ss'}$ will mark the correlation between a pair of relatives one of whom is in class s, the other in class s'; and a reference to the above scheme will at once show us which pair of relatives is being dealt with. Thus $r_{0,12}$ is the correlation of male great-grandchildren with the father of the maternal grandfather, and so on.

(2) Unfortunately the data for great-grandparents is not very extensive. We had altogether only 2996 cases, or about 187 for each of the 16 possible great-grandparental correlations. This is distinctly too few considering the magnitude of the probable errors which occur when we deal with quantities not quantitatively measurable†. Accordingly I paid no attention to the separation of sex in the offspring but grouped my great-grandparental data into four classes: (*a*) those in which there is no change of sex between great-grandparent and offspring, (*b*) those in which there is one change of sex, (*c*) those in which there are two changes, (*d*) those in which

* *Phil. Trans.* A, Vol. 195, p. 102 *et seq.*

† *Phil. Trans.* A, Vol. 195, p. 42.

there are three changes. I further classed Professor Pearson's constants for parental and grandparental data* in the same manner, taking the average of the coefficients obtained by him for each class.

The results are as follows:

Parental Inheritance.

Mean $r_{0,2}$, $r_{1,3}$ = ·5300 ... ... No change of sex.
Mean $r_{0,3}$, $r_{1,2}$ = ·4594 ... ... One change of sex.

Grandparental Inheritance.

Mean $r_{0,4}$, $r_{1,7}$ = ·3697 ... ... No change of sex.
Mean $r_{0,5}$, $r_{1,4}$, $r_{1,6}$, $r_{0,7}$ = ·3004 ... ... One change of sex.
Mean $r_{0,6}$, $r_{1,5}$ = ·2961 ... ... Two changes of sex.

Great-grandparental Inheritance.

†Groups 8 and 0, 15 and 1 give r = ·3471 ... ... No change of sex.
Groups 9 and 0, 11 and 0, 15 and 0, 8 and 1, 12 and 1, 14 and 1 give
r = ·2217 ... ... One change of sex.
Groups 10 and 0, 12 and 0, 14 and 0, 9 and 1, 11 and 1, 13 and 1 give
r = ·1452 ... ... Two changes of sex.
Groups 13 and 1, 10 and 0 give r = ·0375 ... ... Three changes of sex.

The mean result of the four great-grandparental coefficients here found is ·1879, or if all the material be clubbed together and one correlation table formed for it, it is ·1824, the two results being practically identical, considering the probable error of their difference. At the end of this note I give the actual correlation tables, remarking that the classes are those of Mr Galton's eye-colour classification, i.e.

1 = Light blue.	5 = Light brown.
2 = Blue, dark blue.	6 = Brown.
3 = Grey, blue-green.	7 = Dark brown.
4 = Dark grey, hazel.	8 = Very dark brown, black.

For purposes of calculation the fourfold table was formed by classing together 1–3, and 4–8.

Now, although the probable error of these results runs from ·033 to ·055, their invariable decrease with an increasing number of changes of sex is highly significant. It can hardly be said that there is a constant factor of reduction for each change of sex; but, if it existed, a more elaborate system of measurements would probably be required to satisfactorily determine it. Still we can conclude with absolute safety for eye-colour in man and probably therefore for many other characters in other forms of life that:

Every change of sex in the line of ancestry sensibly weakens the intensity of inheritance.

It is desirable accordingly in future heredity experiments to record carefully such changes for they can obviously be of much significance. We may even see in them some justification for refusing to admit inheritance of title through the female line, although the objection would apply equally to the transfer from the female through the male line.

* *Phil. Trans.* A, Vol. 195, p. 106.

† The results here are obtained, not by taking the means of correlation coefficients, but by clubbing groups together.

TABLE I.—*No Change of Sex.*

Offspring.

Great-grandparents.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	Totals
1	—	1	—	—	—	—	—	—	1
2	1	51	35	29	2	15	7	5	145
3	5	33	31	13	—	6	13	—	101
4	—	7	4	9	1	4	3	—	28
5	—	—	—	—	—	—	—	—	—
6	—	7	8	6	3	10	9	—	43
7	—	8	8	7	—	2	8	5	38
8	1	2	4	3	—	—	—	2	12
Totals	7	109	90	67	6	37	40	12	368

Offspring.

Great-grandparents.	*1—3*	*4—8*	Totals
1—3	157	49	206
4—8	90	72	162
Totals	247	121	368

*$h = \cdot44322$, $k = \cdot15042$,

$$\cdot02319r^4 + \cdot13090r^3 + \cdot03333r^2 + r - \cdot35688 = 0,$$

$$r = \cdot3471.$$

TABLE II.—*One Change of Sex.*

Offspring.

Great-grandparents.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	Totals
1	4	3	8	5	—	1	—	—	21
2	8	177	95	76	5	39	31	17	448
3	1	69	85	52	2	20	26	1	256
4	6	30	21	27	2	7	15	1	109
5	—	4	—	—	—	—	—	—	4
6	2	37	27	17	3	30	20	4	140
7	—	15	20	24	3	4	9	9	84
8	—	10	13	12	2	2	7	5	51
Totals	21	345	269	213	17	103	108	37	1113

Offspring.

Great-grandparents.	*1—3*	*4—8*	Totals
1—3	450	275	725
4—8	185	203	388
Totals	635	478	1113

$h = \cdot38909$, $k = \cdot17772$,

$$\cdot02436r^4 + \cdot13697r^3 + \cdot03458r^2 + r - \cdot22496 = 0,$$

$$r = \cdot2217.$$

TABLE III.—*Two Changes of Sex.*

Offspring.

Great-grandparents.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	Totals
1	5	4	9	1	2	—	1	3	25
2	6	175	86	80	4	27	34	12	424
3	3	68	94	57	2	27	27	4	282
4	5	28	21	14	3	5	13	1	90
5	—	4	—	—	—	—	1	—	5
6	1	34	30	34	1	20	14	5	139
7	1	37	34	25	1	16	14	4	132
8	1	4	8	6	—	4	7	2	32
Totals	22	354	282	217	13	99	111	31	1129

Offspring.

Great-grandparents.	*1—3*	*4—8*	Totals
1—3	450	281	731
4—8	208	190	398
Totals	658	471	1129

$h = \cdot37852$, $k = \cdot20911$,

$$\cdot02785r^4 + \cdot13654r^3 + \cdot03958r^2 + r - \cdot14642 = 0,$$

$$r = \cdot1452.$$

* The notation is the same as that used by Pearson, *Phil. Trans.* A, Vol. 195, pp. 1—47.

TABLE IV.

Three Changes of Sex.

Offspring.

Great-grandparents.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	Totals
1	—	3	1	—	—	—	3	1	8
2	1	49	33	18	1	8	11	2	123
3	1	25	28	27	2	10	6	—	99
4	—	14	7	8	—	4	4	—	37
5	—	1	—	—	—	—	2	—	3
6	5	17	8	6	2	3	2	5	48
7	1	24	13	9	—	4	11	2	64
8	—	1	1	—	—	—	1	1	4
Totals	8	134	91	68	5	29	40	11	386

Offspring.

Great-grandparents.	*1—3*	*4—8*	Totals
1—3	141	89	230
4—8	92	64	156
Totals	233	153	386

$h = \cdot 24264, \quad k = \cdot 26275,$

$$\cdot 02290r^4 + \cdot 14603r^3 + \cdot 03188r^2 + r - \cdot 03758 = 0,$$

$r = \cdot 0375.$

TABLE V.

Total Material.

Offspring.

Great-grandparents.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	Totals
1	9	11	18	6	2	1	4	4	55
2	16	452	249	203	12	89	83	36	1140
3	10	195	238	149	6	63	62	5	738
4	1	79	53	58	6	20	35	2	264
5	—	9	—	—	—	—	3	—	12
6	8	95	73	63	9	63	45	14	370
7	2	84	75	65	4	26	42	20	318
8	2	17	26	21	2	6	15	10	99
Totals	58	942	732	565	41	268	299	91	2996

Offspring.

Great-grandparents.	*1—3*	*4—8*	Totals
1—3	1198	735	1933
4—8	534	529	1063
Totals	1732	1264	2996

$h = \cdot 19705, \quad k = \cdot 37238,$

$$\cdot 02591r^4 + \cdot 13798r^3 + \cdot 03669r^2 + r - \cdot 18455 = 0,$$

$r = \cdot 1824.$